Crash-course Topology (拓扑学)

Tuòpū

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Outline

Further leading Literature

- J. Munkres: *Topology*, 2nd edn., 537 pp. (Prentice Hall, 2000), ISBN 0131816292.
- written by the web: *Wikipedia the free encyclopedia*. http://www.wikipedia.org/.

Topological space (拓扑 空间)

Tuòpū

Definition

A topological space *is a set X together with a* topology $\mathcal{O} \subset 2^X$, *i.e. a family of subsets of X such that*

$$\emptyset, X \in \mathcal{O}, \tag{1}$$

$$\forall U_1, U_2 \in \mathcal{O}\colon\ U_1 \cap U_2 \in \mathcal{O}, \tag{2}$$

$$\forall \{U_\alpha\} \subset \mathcal{O}\colon\ \bigcup_\alpha U_\alpha \in \mathcal{O}. \tag{3}$$

The elements $U \in \mathcal{O}$ are called open sets (开集 Kāi jí). A set with $G := X \setminus U$ is called closed set (闭集 Bì jí).

Elementary examples

Example

0. Given an arbitrary set X, the discrete topology (离散 拓扑) is $\mathcal{O}_D := 2^X$, i.e. all sets are open (and thus all sets are closed as well).
1. Given the same set X, the indiscrete topology (非离散拓扑) is $\mathcal{O}_I := \{\emptyset, X\}$, i.e. there are only two open sets, which are at the same time also the only closed sets.
2. Given the real line $\mathbb{R}$, then its standard topology defines $U \subset \mathbb{R}$ as open iff for every $x \in U$, there is an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. (a, b) is called an open interval (开区间) while $[a, b]$ (i.e. including the endpoints) is called a closed interval.

   Hint: The interval $[a, b)$ is neither open nor closed.

More about open and closed sets

Exercise

Show that, given a set X and defining $U \subset X$ open iff $G := X \setminus U$ is closed, then the axioms of topology are equivalent to the following axioms about the family $\mathcal{G} \subset 2^X$ of closed sets:

$$\emptyset, X \in \mathcal{G},$$
$$\forall G_1, G_2 \in \mathcal{G}:\ G_1 \cup G_2 \in \mathcal{G},$$
$$\forall \{G_\alpha\} \subset \mathcal{G}:\ \bigcap_\alpha G_\alpha \in \mathcal{G}.$$

Hint: de Morgan's laws: $G_1 \cup G_2 = X \setminus ((X \setminus G_1) \cap (X \setminus G_2))$

Metric spaces (度量空间) I

Definition

A metric space is a set X together with a metric (distance function, 距 (Jù) 离函数) d, i.e. a map $d\colon X \times X \to [0, \infty)$ such that

$$\begin{aligned} d(y,x) &= d(x,y), \\ d(x,y) &= 0 \text{ iff } x = y, \\ d(x,z) &\le d(x,y) + d(y,z) \end{aligned}$$

for all $x, y, z \in X$.

Metric spaces (度量空间) II

Definition

Given a metric space (X, d) and a point $x \in X$. The open ball (开球体) in X around x with radius $r \geq 0$ is

$$B_r(x) := \{y \in X : d(x, y) < r\}.$$

The topology generated by (产 Chǎn 生) the metric is the family $\mathcal{O}_d$ of sets $U \subset X$ such that for every $x \in U$, there is an $\epsilon > 0$ with $B_\epsilon(x) \subset U$.

Exercise

It is easy to check the axioms for the topology $\mathcal{O}_d$.

Metric spaces (度量空间) III

Example

1. Given an arbitrary set X, then $\mathcal{O}_D$ the discrete topology is generated by the discrete metric (离散 [sàn] 度量), i.e.

$$d_0(x, y) = \begin{cases} 1 & \text{iff } x \neq y, \\ 0 & \text{iff } x = y. \end{cases}$$

2. The indiscrete topology in general cannot be generated by any metric, but this is a bit harder to prove.
3. The standard topology on $\mathbb{R}$ comes from the metric $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$ where $|\cdot|$ denotes the absolute value (绝 [Jué] 对值 [zhí]).

Continuous maps (连续映射)

xù yìngshè

Remark

Remember the ϵ-δ-definition of continuity, i.e. a map $f\colon I \to \mathbb{R}$ is continuous iff for every $x \in I$ and $\epsilon > 0$ there is a $\delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$.

Definition

Given two topological spaces $(X, \mathcal{O})$ and $(Y, \mathcal{P})$, we say that a map $f\colon X \to Y$ is continuous iff for all $U \in \mathcal{P}$ we have $f^{-1}(U) \in \mathcal{O}$.

Theorem

Given two metric spaces (X, d) and (Y, ρ), then $f\colon X \to Y$ is continuous in the sense of metric spaces, iff f is continuous in the induced topologies.

Proof.

Let us first assume that $f\colon X \to Y$ is continuous in the sense of metric spaces. Let now $U \subset Y$ be open. We wish to show that $V := f^{-1}(U) := \{x \in X : f(x) \in U\}$ is open. Let thus $x \in V$ and $y := f(x) \in U$. Since U is open there is an $\epsilon > 0$ such that $B_\epsilon(y) \subset U$. Moreover since f is continuous, there is a $\delta > 0$ such that for all $x' \in B_\delta(x)$, $f(x') \in B_\epsilon(y) \subset U$. But this implies that $B_\delta(x) \subset V = f^{-1}(U)$. Therefore $V = f^{-1}(U)$ is open. This shows one direction of the claim.

For the other direction assume now that $f\colon X \to Y$ is continuous in the sense of topological spaces. Given any $x \in X$ and $\epsilon > 0$, we wish to find a $\delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$. Consider therefore the open set $U := B_\epsilon(f(x)) \subset Y$. Since f is continuous in the sense of topological spaces, we know that $V := f^{-1}(U) \subset X$ is an open set. Moreover it contains x. But therefore there exists a $\delta > 0$ such that $B_\delta(x) \subset V$. This implies that $f(B_\delta(x)) \subset U$ and therefore δ fulfills the requirements of the definition of continuity in the sense of metric spaces. This completes the proof. □

Example (continuous maps)

0. The identity map on a space with one topology, i.e. $\mathrm{id}\colon (X,\mathcal{O}) \to (X,\mathcal{O})$. This is indeed an homeomorphism (bijective (双射 shè), bi-continuous (双连续) map).
1. coarsening (粗化 Cū) of topology, e.g. $(X,\mathcal{O}) \to (X,\mathcal{O}_I) : x \mapsto x$ where the latter is the same set X endowed with the indiscrete topology. Also any map $(X,\mathcal{O}_D) \to (Y,\mathcal{P})$ where $\mathcal{O}_D$ is the discrete topology.
2. the usual continuous real functions $f\colon I \to \mathbb{R}$ where I is a (connected) interval.
3. emeddings (嵌入 Qiàn), i.e. $e\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O})$ where $X \subset Y$ is an open or closed subset and $\mathcal{O}_X := \{U \cap X : U \in \mathcal{O}\}$.
4. projections (投影 tóu), i.e. $p\colon (X,\mathcal{O}) \to (Y,\mathcal{O}_p)$ where $\mathcal{O}_p$ is the topology generated by $\{p(U) : U \in \mathcal{O}\}$. We call $(Y,\mathcal{O}_p)$ the factor space (因素 sù 空间) if p is surjective (满的), and $\mathcal{O}_p$ the factor topology.

Meaning of topology

Q: What are the topological properties (性质 zhì) of a space?
a: The things that are invariant under homeomorphisms (同胚, bijective, bi-continuous maps).

Example

1. The (round) circle 'O' (with the usual topology[1]) is homeomorphic to the square '□'.
2. With an analogous argument also the figure '8' is homeomorphic to the Letter 'B'.

[1] as an emedded subset of $\mathbb{R}^2$

Connectedness (连通)

Q: what about the colon ':' and the capital letter 'I'?
a: The latter is connected, but the former is not.

Definition

Given a topological space $(X, \mathcal{O})$ we say that X is disconnected (非连通) iff it can be written as the disjoint union (不相交) of two nonempty (非空) open sets, i.e. $X = U_1 \cup U_2$, $x_i \in U_i \in \mathcal{O}$ and $U_1 \cap U_2 = \emptyset$.
A topological space is called connected iff it is not disconnected.

zǔ jiàn

The connected components (连接组件) of a topological space are the maximal (最大) connected open subsets.

Remark

A connected space has exactly one connected component. A disconnected space has more than one.

Path-connected (路径 连接)

jìng

Definition

Given a topological space $(X, \mathcal{O})$ it is said to be path-connected iff for every pair of points $x, y \in X$ there is a path, i.e. a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.
A topological space is called path-disconnected iff it is not path-connected.

We can also introduce path-connected components.

Example (connected and disconnected)

1. The interval (in standard topology) is connected, because given a disjoint union $[0,1] = I = U_0 \cup U_1$ with $x_i \in U_i$ for $i = 0, 1$, then there is at least one boundary point (边界点) $x \in I$, i.e. a point such that every open subset $x \in U_x \subset I$ contains a point in U_0 and a point in U_1.[2] But then x is either in U_0 or in U_1 but at least one. Without loss of generality in U_0. Therefore U_0 cannot be open. Which contradicts the assumption.
   The interval is path-connected, because $\gamma = \text{id}$ is continuous.
2. The space $(-1,0) \cup (0,1) \subset \mathbb{R}$ is neither connected nor path-connected. Namely it has 2 connected components: $(-1,0)$ and $(0,1)$.

Q: Are there examples where the notions differ?

[2] The boundary points of a nonempty interval are its endpoints. Every open

Relation between connected and path-connected

Q: So what is the relation?

Proposition

Given a topological space X which is path-connected, then it is also connected.

Proof.

Assume that $(X, \mathcal{O})$ is path-connected but not connected. Then we can write $X = U_1 \cup U_2$ with disjoint nonempty open subsets U_i, $i = 1, 2$. Let $x_i \in U_i$ and $\gamma\colon I \to X$ be a path that connects x_1 with x_2. Since γ is continuous, $V_i := \gamma^{-1}(U_i) \subset I$ are nonempty disjoint open subsets of I. But that is a contradiction, because the interval I is connected. Therefore the assumption is wrong and thus the statement true. □

Locally connected (本地连接)

Definition

Given a topological space X. It is said to be locally connected iff every point $x \in X$ and every open neighborhood $x \in V_x \subset X$ contains a connected open neighborhood $x \in U_x \subset V_x$.
The analogous definition for locally path-connected.

Example

The $\{y\text{-axis}\} \cup \{(x, \sin 1/x)\}$-curve can be made path-connected (by adding a curve) while still not being locally path-connected.

Compactness (紧凑)

Jǐncòu

Q: What is the intrinsic (固有) difference between the open interval $(0, 1)$ and the closed interval $[0, 1]$ as topological spaces?

Gù

Hint: Ignore the embedding, both are closed and open sets in themselves.

a: One is compact the other is not.

Definition

A topological space $(X, \mathcal{O})$ is said to be compact iff every open cover (开覆盖) $X = \bigcup_\alpha U_\alpha$ contains a finite subcover (有限子覆盖), i.e. there is an $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n$ such that $X = \bigcup_{i=1}^n U_{\alpha_i}$.

Fùgài xiàn

Example

0. Finite sets are compact, because all their covers are finite.
1. The open interval $(0, 1) \subset \mathbb{R}$ is not compact, e.g. every subcover of the cover by $U_\alpha := (0, 1 - \frac{1}{\alpha})$, $\alpha \in \mathbb{N}^*$ requires infinitely many U_α.

Subspaces of compact spaces

Proposition

Every closed subset of a compact space is compact.

Proof.

Let $(X, \mathcal{O})$ be compact and $Y \subset X$ be closed. Let an open cover $\{U_\alpha\}$ of Y be given. Remember that the subspace topology $\mathcal{O}_Y$ is $U_\alpha \subset Y$ is open iff there is an $V_\alpha \subset X$ that is open and $U_\alpha = Y \cap V_\alpha$. Therefore the $\{V_\alpha\}$ together with $V_0 := X \setminus Y$ cover X. Since X is compact there is an $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n$ such that $V_{\alpha_1}, \dots, V_{\alpha_n}$ cover X. If one of them is $X \setminus Y$, discard it. Then the $U_{\alpha_1}, \dots, U_{\alpha_n}$ are a finite cover of Y. □

Q: Is the converse also true?

non-Hausdorff (非豪斯多夫) problem

Háo sī

Problem: Consider the space $X = \{0, 1\}$, $\mathcal{O} = \{\emptyset, X\}$ and $Y = \{0\} \subset X$. Clearly X and Y are compact. However $\mathcal{G} = \{\emptyset, X\}$, so Y is not closed.

Solution: The space $(X, \mathcal{O})$ is particularly bad, e.g. there are no disjoint open subsets around 0 and 1 (even though these points are not the same).

Definition

Given a topological space $(X, \mathcal{O})$. It is said to be Hausdorff iff for every pair of different points $x_i \in X$, $i = 1, 2$ there are disjoint open neighborhoods $x_i \in U_i \subset X$, $U_1 \cap U_2 = \emptyset$.

Example

Metric spaces (X, d) are Hausdorff, e.g. for $x_1, x_2 \in X$, the open balls $B_\epsilon(x_i)$, $i = 1, 2$ are disjoint for $\epsilon = \frac{1}{3} d(x_1, x_2)$.

Images of compact spaces

Proposition

A continuous map applied to a compact space gives a compact subspace. It is moreover closed if the target space is Hausdorff.

Proof.

Let $f\colon X \to Y$ be continuous and $(X, \mathcal{O})$ be compact. Any open cover $\{U_\alpha\}$ of $f(X) \subset Y$ gives an open cover of X as $\{V_\alpha := f^{-1}(U_\alpha)\}$. Since X is compact there is a finite subcover $V_{\alpha_1}, \dots, V_{\alpha_n}$. But then $U_{\alpha_1}, \dots, U_{\alpha_n}$ is a finite subcover of $f(X)$. This proves the first part.
Let in addition Y be Hausdorff. We need to show that $U_0 := Y \setminus f(X) \subset Y$ is open. Let therefore $y \in U_0$. We need to find an open neighborhood of $y \in U_y \subset U_0$. For each $y_\alpha \in f(X) \subset Y$ there are disjoint open neighborhoods $y_\alpha \in U_\alpha \subset Y$ and $y \in W_\alpha \subset Y$.[3] The $\{U_\alpha\}$ form an open cover of $f(X)$ which is compact. Therefore the finite union $U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$ also covers $f(X)$. It is disjoint from the set $U_y := W_{\alpha_1} \cap \dots \cap W_{\alpha_n}$, which is a finite intersection of open sets and thus open. This completes the proof, because U_0 is the union of open neighborhoods. □

Compact subsets of $\mathbb{R}^n$

Theorem (Heine–Borel)

A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.[4]

Corollary (Extreme value thm)

Given a continuous function $f\colon X \to \mathbb{R}$ on a compact space $(X, \mathcal{O})$, then f attains its infimum (下确界, què) and its supremum (上确界) on X.

Proof.

Let $R := f(X) \subset \mathbb{R}$ be the range (值域, Zhí yù) of the function. This is a compact set and thus the disjoint union of finitely many closed intervals. The minimum (最低) / maximum (最高) of the interval bounds is the minimum/ maximum of the function and in particular in the range of the function. □

Remark

Consider the set $\{\frac{1}{n} : n \in \mathbb{N}^*\}$. Its infimum is 0, but that is not an element of the set.

[4] in any metric d

Local compactness

Definition

A topological space $(X, \mathcal{O})$ *is called locally compact iff for every point* $x \in X$ *and every open neighborhood* $x \in U \subset X$*, there is a smaller open neighborhood* $x \in V$ *whose closure* $\bar{V}$ *is compact and contained in* U*.*

Example

subsets of $\mathbb{R}^n$ are locally compact, because every open subset $x \in U$ contains a bounded subset $B_\epsilon(x) \subset U$, and then $\bar{B}_{\epsilon/2}(x) \subset B_\epsilon(x) \subset U$ is compact and strictly contained.

Compactification I

Proposition

Given a locally compact Hausdorff space $(X, \mathcal{O})$, we can embed it into a compact space $\bar{X} = X \dot{\cup} \{\infty\}$ with topology $\bar{\mathcal{O}}$ that is defined as open neighborhoods of ∞ are the complements of compact subsets of X. In particular $\bar{X}$ is Hausdorff.

Compactification II

Proof.

To see that $\bar{X}$ is compact, note that every open cover must contain an open neighborhood of ∞. But the complement is compact and thus covered by a finite selection from the cover. This proves compactness. Also the embedding property is obvious, because $\mathcal{O} \subset \bar{\mathcal{O}}$. To see that $\bar{X}$ is Hausdorff, note that two points from X are separated by the disjoint open neighborhoods in X. Finally $x \in X$ is separated from ∞, because there is an open neighborhood $x \in U_x \subset X \subset \bar{X}$ whose closure is compact and therefore the closure's complement U_∞ is open and disjoint to U_x. This completes the proof. □

Compactification III

Remark

It is also possible to ask for the following universality property:
The Stone–Čech compactification is an embedding $c\colon X \to \bar{X}$[5] into a compact space $\bar{X}$ such that every continuous map $f\colon X \to C$ into any compact space C extends uniquely to a continuous map $\tilde{f}\colon \bar{X} \to C$.
In particular every bounded continuous function $f\colon X \to \mathbb{R}$ extends uniquely to a continuous function on $\bar{X}$.
However these compactifications are in general much bigger than X, except when X is already compact (because then $X = \bar{X}$).

[5] Not necessarily the 1-point compactification

Limits & Separation problem

Q1: How to define limit (jíxiàn 极限) in a topological space?

Definition

A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ *has the limit* $x \in X$ *in a topological space* X *(*$x_n \to x$ *as* $n \to \infty$*) iff for every open neighborhood* $x \in U \subset X$*, there is an* N *such that* $n \geq N$ *implies* $x_n \in U$.[6]

Q2: Are limits unique?

Example

1. Consider the indiscrete topology $\mathcal{O}_I = \{\emptyset, X\}$ where X contains more than one point. Then every sequence converges towards every point in X.
2. Consider on the other hand metric spaces (X, d). Then the topological and metric notion of convergence coincide and ensure a unique limit (if it exists).

[6] It is also possible to generalize to converging nets (网).

Separation Axioms

Definition

Given a topological space $(X, \mathcal{O})$ it is called

0. *T0 iff for every pair of points $x, y \in X$, there is an open set $U \subset X$ containing exactly one of the points;*
1. *T1 iff for every pair of points $x_i \in X$, $i = 1, 2$, there are open sets $x_i \in U_i \subset X$ that do not contain the other point;*
2. *Hausdorff iff for every pair of points $x_i \in X$, $i = 1, 2$, there are disjoint open sets $x_i \in U_i \subset X$, $U_1 \cap U_2 = \emptyset$;*
3. *regular iff for every pair of a point $x \in X$ and a closed set $G \subset X$ that does not contain x, then there are disjoint open sets $x \in U_x \subset X$ and $G \subset U \subset X$, $U_x \cap U = \emptyset$;*
4. *normal iff for every pair of compact disjoint sets $G, H \subset X$ there are disjoint open neighborhoods $G \subset U_G \subset X$ and $H \subset U_H \subset X$, $U_G \cap U_H = \emptyset$.*

Example (separation axioms)

1. Metric spaces have all these properties. To see T4, you build δ-parallel neighborhoods $G_\delta := \{x \in X : d(x, G) < \delta\}$ with $\delta = \frac{1}{3} d(G, H)$ where $d(x, G) := \inf_{g \in G} d(x, g)$ is indeed a minimum and thus 0 iff $x \in G$ and correspondingly for $d(G, H) := \inf_{h \in H} d(h, G)$.
2. The indiscrete topology is not even T0.

The Hausdorff property asserts uniqueness of a limit (if it exists).

Proposition

T2 $\Rightarrow$ T1 $\Rightarrow$ T0
Assuming that the one-point sets $\{x\} \subset X$ are closed sets, then T4 $\Rightarrow$ T3 $\Rightarrow$ T2.

Countability property (可数)

Proposition

Given an open subset $U \subset \mathbb{R}^n$, then there is a countable open cover.

Proof.

Consider the points with only rational coordinates. Since the rational numbers are countable, we can also count these points. Consider futher the open balls around rational points with positive rational radii. These are still countable.
Finally note, that for every point $x \in \mathbb{R}^n$ and every open ball $x \in B \subset \mathbb{R}^n$ there is a rational point $q \in B$ and a positive rational number $r > 0$ such that $x \in B_r(q) \subset B$. □

First and second countable

Definition

A topological space $(X, \mathcal{O})$ is said to have a countable basis $\mathcal{B} \subset \mathcal{O}$ at $x \in X$ iff every $B \in \mathcal{B}$ has $x \in B$ and every $x \in U \subset X$ has some $B \in \mathcal{B}$ with $B \subset U$. A space that has at every of its points a countable basis is called first countable.

Metric spaces are first countable, via balls with rational radii.

Definition

A topological space $(X, \mathcal{O})$ is said to be second countable iff it has a countable topological basis $\mathcal{B} \subset \mathcal{O}$, i.e. for every open set $U \subset X$ and every point $x \in U$ there is an $x \in B \in \mathcal{B}$ with $B \subset U$.

Remark

Metric spaces are second countable iff they are separable (可分), i.e. there is a countable set $\{x_n\}_{n=1}^{\infty} \subset X$ such that in every nonempty open set $U \subset X$ there is a point $x_n \in U$.

Sequentially compactness (序列 紧凑) I

Xùliè

An alternate notion of compactness is the following:

Definition

A topological space $(X, \mathcal{O})$ is said to be sequentially compact iff every infinite sequence $\{x_n\}_{n=1}^{\infty} \subset X$, has a converging subsequence, i.e. there are $n_1 < n_2 < n_3 < \dots$ and an $x \in X$ such that $x_{n_i} \to x$ as $i \to \infty$.

Sequentially compactness (序列 紧凑) II

Xùliè

Example

1. The space $[0,1]^{\mathbb{R}}$ is compact (due to Tychonov's theorem), but not sequentially compact, ...
2. subsets of $\mathbb{R}^n$ are compact iff they are sequentially compact.
3. There are infinite topological spaces that are sequentially compact and non-compact. They do however not have a metric topology.

Proposition

Given a first countable topological space X, then its compactness implies sequentially compactness.

Sequentially compactness (序列 紧凑) III

Xùliè

Proof.

Assume X is compact, but not sequentially compact. In particular, there is an infinite sequence $\{x_n\}$ wihtout any accumulation point, i.e. for all $y \in X$ there is an open neighborhood $y \in U_y \subset X$ such that $\{x_n\} \cap U_y$ is finite, because if in the countable topological base $\{U_i\}$ of a point $x \in X$ each U_i contains infinitely many x_n, then we can filter them to form infinitely many x_{n_j} in every U_i with $j > i$ and thus have a sequence converging to x. On the other hand $\{x_n\} = \bigcup_{y \in X} \{x_n\} \cap U_y$ and due to compactness of X, there are finitely many $y_1, \ldots, y_N$ such that $\bigcup_{k=1}^{N} U_{y_k} = X$. But this would imply that $\{x_n\}$ is finite – a contradiction. □

Metrizability (可度量)

A topological space $(X, \mathcal{O})$ is called metrizable iff there exists a metric d on X that induces its topology.

Theorem (Urysohn (乌雷松) lemma)

Given a normal space (T4) X and two disjoint closed sets $A, B \subset X$ and $a < b$. Then there exists a continuous function $f\colon X \to [a, b]$ such that $f(A) = \{a\}$ and $f(B) = \{b\}$.

Corollary (Urysohn theorem)

Every regular space (T3) with a countable topological basis (second countable) is metrizable.

Idea of the proof.

You embed the space X into the Hilbert cube $[0, 1]^{\mathbb{N}}$ with normalized l_2-metric, i.e.

$$d_2(\mathbf{x}, \mathbf{y}) := 1 - 1 \Bigg/ \left(1 + \sqrt{\sum_{i=0}^{\infty} (x_i - y_i)^2}\right).$$

□

Product of topological spaces

Example

The topology of $\mathbb{R}^2$ can also be generated by open rectangles $\{(a,b)\times(c,d) : a<b,\ c<d\}$.

Definition

Given a family of topological spaces $\{(X_\alpha,\mathcal{O}_\alpha) : \alpha\in A\}$, then their product is the set $X := \prod_{\alpha\in A} X_\alpha$ of "tuples" $x\colon A\to\bigcup_{\alpha\in A} X_\alpha$ such that $x_\alpha\in X_\alpha$ for every $\alpha\in A$. This is endowed with the topology generated by cylinders $\pi_\alpha^{-1}(\mathcal{O}_\alpha)$ where $\pi_\alpha\colon X\to X_\alpha : x\mapsto x_\alpha$ is the projection onto the α-component.

Example

$\mathbb{R}^n = \mathbb{R}\times\cdots\times\mathbb{R}$ with n factors, because $(a,b)\times(c,d) = \pi_1^{-1}(a,b)\cap\pi_2^{-1}(c,d)$.

Tychonoff's compactness theorem (吉洪诺夫 定理) I

Jí hóng nuò

Theorem (Tychonoff)

Given a family $\{(X_\alpha, \mathcal{O}_\alpha)\}$ of compact spaces, then their product $\prod_\alpha (X_\alpha, \mathcal{O}_\alpha)$ is also compact.

Idea of proof.

Tychonoff's compactness theorem (吉洪诺夫 定理) II

Jí hóng nuò

1. A collection $\mathcal{B}$ of subsets $G \subset X$ is said to have the finite intersection property iff every finite choice of subsets $G_1, \ldots, G_n \in \mathcal{B}$ has nonempty intersection. A topological space X is compact iff for every collection $\mathcal{B}$ of subsets $G \subset X$ that has the finite intersection property, the joint intersection $\bigcap_{G \in \mathcal{B}} G$ is nonempty.
2. Among the family of collections $\mathcal{B}$ with the finite intersection property there are maximal collections $\mathcal{D}$ with that property.
3. $\mathcal{D}_\alpha := \pi_\alpha(\mathcal{D}) := \{\pi_\alpha(D) : D \in \mathcal{D}\}$ is a collection of subsets of X_α with the finite intersection property. Since X_α is compact, there is an $x_\alpha \in \bigcap_{D \in \mathcal{D}_\alpha} D$. But then $\mathbf{x} := (x_\alpha)_{\alpha \in A}$ is an element in $\bigcap_{D \in \mathcal{D}} D$ which completes the proof.

Tube lemma (管引理)

Guǎn Yǐn

Lemma

Given two topological spaces X and Y, where Y is compact. If the open set $N \subset X \times Y$ contains a slice (切片) $x_0 \times Y$, then N also contains a tube $W \times Y$ for some open neighborhood $x_0 \in W \subset X$.

Qiēpiàn

Proof.

Since $x_0 \times Y \subset W \subset X \times Y$ there are $U_\alpha \subset X$ and $V_\alpha \subset Y$ such that $U_\alpha \times V_\alpha \subset N$ and $x_0 \times Y \subset \bigcup_\alpha U_\alpha \times V_\alpha$. Since $x_0 \times Y \approx Y$ is compact, finitely many of them $\{U_i \times V_i\}_{i=1}^n$ are sufficient. Discarding those pairs for which $x_0 \notin U_i$, we can ensure that x_0 lies in

$$W := U_1 \cap \cdots \cap U_n.$$

But as the intersection of finitely many open sets $W \subset X$ is open. This completes the proof. □

Point-open topology (点开拓扑)

Suppose we have an index set X together with identical topological spaces $(Y, \mathcal{P})$, then the product topology on $Y^X := \prod_{x \in X} Y := \{f \colon X \to Y\}$ is generated by the open sets $O := \{f \in Y^X : f(x_1) \in U_1, \ldots, f(x_n) \in U_n\}$ for $n \in \mathbb{N}$, $\{x_i\} \subset X$, and $U_i \subset Y$ open sets.

Definition

Given a set X together with a topological space Y, then the point-open topology on Y^X is the product topology.

Example

Given $X = \mathbb{R} = Y$, then an open set through (x_1, U_1),...,(x_n, U_n) are these functions that pass through U_1 at x_1,..., and through U_n at x_n.

The use of the point-open topology is the following:

Proposition

A sequence of functions $\{f_n \colon X \to Y\}$ converges pointwise to $f \colon X \to Y$ iff it converges in the point-open topology.

Uniform topology (制服 拓扑)

Zhìfú

The problem is that convergence in the above topology does not imply any additional properties of the limit function f. To obtain nice functions, we define the following.

Definition

Given a set X together with a metric space (Y, d), the bounded functions are $B(X, Y) := \{(f\colon X \to Y) : d(f(X)) < \infty\}$ where $d(U) := \sup_{y_1, y_2 \in U} d(y_1, y_2)$ and $d(\emptyset) := 0$. The sup-metric is $\bar{d}\colon B(X, Y) \times B(X, Y) \to [0, \infty)$ which for every pair $f, g \in B(X, Y)$ is defined as

$$\bar{d}(f, g) := \sup_{x \in X} d(f(x), g(x)).$$

The uniform topology is the one on $B(X, Y)$ generated by the sup-metric.

Proposition

Uniform convergence of bounded continuous functions implies continuity of the limit function.

This is a standard exercise in real analysis.

Compact-uniform topology (紧凑制服拓扑) I

Q: Can we weaken the restriction on the sequence $\{f_n\} \subset \mathrm{C}(X, Y)$?
a: Yes, in the following situation.

Definition

1. *A topological space X is called compactly generated iff every set $A \subset X$ is open whenever $A \cap C \subset C$ is open for every compact subset $C \subset X$.*
2. *Given a compactly generated topological space X and a metric space (Y, d), then the compact-uniform topology on $\mathrm{C}(X, Y)$ is generated by the open sets $O(C, f_0, \epsilon) := \{f \in \mathrm{C}(X, Y) : \bar{d}(f|C, f_0) < \epsilon\}$ for every compact subset $C \subset X$, continuous map $f_0 \in \mathrm{C}(C, Y)$, and every $\epsilon > 0$.*

Compact-uniform topology (紧凑制服拓扑) II

Lemma

Locally compact spaces as well as first countable spaces are compactly generated. □

Example

The spaces $\mathbb{R}^n$, their subsets, as well as metric spaces are compactly generated.

Lemma

A map $f: X \to Y$ on a compactly generated topological space X is continuous iff $f|C$ is continuous for all compact $C \subset X$. □

Compact-uniform topology (紧凑制服拓扑) III

We can now generalize the convergence condition for continuous maps as follows.

Theorem

Given a compactly generated topological space X, a metric space (Y, d), and a family of continuous functions $\{f_n\}_{n=1}^{\infty} \subset \mathrm{C}(X, Y)$ that converges in compact-uniform topology to a function $f\colon X \to Y$, then $f \in \mathrm{C}(X, Y)$ is continuous.

Proof.

Applying the uniform convergence theorem to the sequence $\{f_n\}$ over every compact subset $C \subset X$ of X, we see that $f|C$ is continuous. But due to the last lemma, $f|X$ must then also be continuous. This completes the proof. □

The theorem states that $\mathrm{C}(X, Y) \subset Y^X$ is a closed set in the compact-uniform topology.

Compact-uniform topology (紧凑制服拓扑) IV

Example

Consider the exponential function $\exp\colon \mathbb{R} \to (0, \infty)$. We know that its Taylor series is $(T\exp)(x) = \sum_{n\geq 0} \frac{1}{n!}x^n$. This does not converge uniformly on the whole real line. It does however converge uniformly on compact subsets, e.g. closed intervals $[a, b]$. Therefore the exponential function is continuous (actually it is even analytic).

Compact-open topology (紧致开拓扑) I

The evaluation map is an example of a higher order function that requires infinite dimensional spaces:
$\mathrm{ev}\colon X \times Y^X \to Y : (x, f) \mapsto f(x)$ where $x \in X$ and $f\colon X \to Y$ is any map. The question is what topology can we put on $\mathrm{C}(X, Y) \subset Y^X$ such that $\mathrm{ev}\,|X \times \mathrm{C}(X, Y)$ is continuous.

Definition

Given a compactly generated topological space X and a topological space Y, then $\mathrm{C}(X, Y)$ the space of continuous maps from X to Y is endowed with the compact-open-topology, i.e. the topology is generated by the open sets $O(C, U) := \{f \in \mathrm{C}(X, Y) : f(C) \subset U\}$ for every compact subset $C \subset X$ and every open subset $U \subset Y$.

Analogous to the continuity property for metric spaces, we can also prove the following equivalence of topologies.

Compact-open topology (紧致开拓扑) II

Proposition

Given a compactly generated topological space X and a metric space (Y, d), then the compact-open topology coincides with the compact-uniform topology. □

Proposition

Given a locally compact Hausdorff space X and a topological space Y, then $\mathrm{ev}\colon X \times \mathrm{C}(X, Y) \to Y : (x, f) \mapsto f(x)$ is continuous.

Compact-open topology (紧致开拓扑) III

Proof.

Given a point $(x, f) \in X \times C(X, Y)$ and an $f(x) = \mathrm{ev}(x, f) \in V \subset Y$ open, we wish to find an open neighborhood $(x, f) \in W \subset X \times C(X, Y)$ that is mapped $\mathrm{ev}(W) \subset V$. Since $f\colon X \to Y$ is continuous, there is an open neighborhood $x \in U \subset X$ that is mapped $f(U) \subset V$. By reducing U in size we can make $\bar{U} \subset X$ compact and still $f(\bar{U}) \subset V$. Consider the set $W := U \times C(\bar{U}, V) \subset X \times C(X, Y)$. It is open and contains the point (x, f). But moreover $(x', f') \in W$ is mapped $\mathrm{ev}(x', f') = f'(x') \in V$. This completes the proof. □

Currying (柯里化) I

Let X_i, $i = 1, 2$ and Y be topological spaces. Given a map $f\colon X_1 \times X_2 \to Y$ we can construct for every $x_2 \in X_2$ a map $f_{x_2} : X_1 \to Y : x_1 \mapsto f(x_1, x_2)$ and combine them as $F\colon X_2 \to Y^{X_1} : x_2 \mapsto f_{x_2}$. The question is now how continuity behaves under this construction.

Theorem

Given a locally compact Hausdorff topological space X_1 and two topological spaces X_2 and Y. The joint function $f\colon X_1 \times X_2 \to Y$ is continuous iff $F\colon X_2 \to \mathrm{C}(X_1, Y) : x_2 \mapsto f_{x_2}$ is continuous in the compact-open topology.

Proof.

Currying (柯里化) II

Suppose first that F is continuous. Then f is continuous because it is the composition of continuous maps

$$X_1 \times X_2 \xrightarrow{\mathrm{id}_{X_1} \times F} X_1 \times \mathrm{C}(X_1, Y) \xrightarrow{\mathrm{ev}} Y.$$

Conversely, let f be continuous. To show continuity of F, we take a generator $O(C, U)$ of the compact-open topology on $\mathrm{C}(X_1, Y)$ that contains $F(x_2)$. It is sufficient to find a neighborhood $x_2 \in W \subset X_2$ that maps into $O(C, U)$. $F(x_2) \in O(C, U)$ means $f(x_1, x_2) \in U$ for every $x_1 \in C$, i.e. $f(C \times x_2) \subset U$. Continuity of f implies that $f^{-1}(U) \subset X_1 \times X_2$ is open and contains the subset $C \times x_2$. Then $f^{-1}(U) \cap C \times X_2 \subset C \times X_2$ is also open. The tube lemma 2 implies that there is a neighborhood $x_2 \in W \subset X_2$ such that $C \times W \subset f^{-1}(U)$. But then $x_1 \in C$ and $x_2 \in W$ imply $f(x_1, x_2) \in U$. Therefore $F(W) \subset O(C, U)$ as desired. □

Equicontinuous (等度连续)

Definition

Given a topological space X and a metric space (Y, ρ), a family of functions $\{f_\alpha\} \subset Y^X$ is said to be equicontinuous iff for every $x \in X$ and every $\epsilon > 0$, there is an open neighborhood $x \in U_x \subset X$ such that $x' \in U_x$ implies

$$\rho(f_\alpha(x'), f_\alpha(x)) < \epsilon$$

for all α.

Note that all functions f_α are continuous, but moreover the continuity relation does not depend on the particular α.

Arzelà–Ascoli Theorem (阿尔泽拉–阿斯科利定理)

Theorem (Arzelà–Ascoli)

Given a locally compact Hausdorff topological space X and a metric space (Y, ρ) together with a family $\mathcal{F} = \{f_\alpha : X \to Y\} \subset \mathrm{C}(X, Y)$ of continuous functions. Then $\mathcal{F}$ is equicontinuous and for every $x \in X$, $F_x := \{f(x) : f \in \mathcal{F}\}$ is contained in a compact subset of Y iff $\mathcal{F}$ is contained in a compact subspace of $\mathrm{C}(X, Y)$.

Easier part of the proof.

Let $\mathcal{F}$ be equicontinuous and for every $x \in X$, $F_x \subset C_x \subset Y$ for compact C_x. Let $G \subset Y^X$ be the closure of $\mathcal{F}$ in Y^X. We want to show that G is compact. It is contained in the space $\prod_{x \in X} C_x$ which is compact in product topology. It is moreover closed, because the C_x are closed. Therefore G is compact (independent of the embedding). Every function $f \in G$ is continuous, because $\mathcal{F} \subset \mathrm{C}(X, Y) \subset Y^X$ and the latter is closed. Therefore $\mathcal{F} \subset G \subset \mathrm{C}(X, Y)$ as required. (Note that we did not require the

Application of the Arzela–Ascoli Theorem

Theorem (Function theory)

Let $\{f_n\}_{n=1}^{\infty} \subset \mathrm{C}^{\omega}(C)$ be a sequence of holomorphic functions on a compact subset $C \subset \mathbb{C}$. Suppose that $f_n \to f$ converges uniformly, then $f \in \mathrm{C}^{\omega}(C)$ is also holomorphic and for every $k \in \mathbb{N}$ there is a subsequence that converges uniformly in the C^k-norm.

Problems with naive dimension theory (朴素维 度合论)

Púsù wéi

Problem 1: There is a bijection from $\mathbb{R}^2$ to $\mathbb{R}$, e.g. the numbers $(a_m a_{m-1} \dots a_0.a_{-1}a_{-2}\dots, b_n b_{n-1}\dots b_0.b_{-1}b_{-2}\dots)$ are mapped to $a_m 0 a_{m-1} 0 \dots a_n b_n \dots a_0 b_0.a_{-1} b_{-1} \dots$. where $a_i, b_i \in \{0, \dots, 9\}$. Fortunately this map is not continuous.

Problem 2: There is even a surjective continuous map $d\colon [0,1] \to [0,1]^2$, e.g. the dragon curve. Fortunately this map is not injective.

Remark

Lì pǔ xī cí

Lipschitz continuous (利普希茨 连续) maps do not increase dimension.

Lebesgue covering dimension (勒贝格 覆盖维度) I

Lēi bèi gé

Definition

1. *A collection $\mathcal{A} \subset \mathcal{O}$ of open subsets in some topological space $(X, \mathcal{O})$ is said to have order $m \in \mathbb{N}$ iff there are m different elements $A_1, \ldots, A_m \in \mathcal{A}$ such that $x \in A_1 \cap \cdots \cap A_m$ and no point $x \in X$ lies in more than m open sets of $\mathcal{A}$.*
2. *A topological space $(X, \mathcal{O})$ is said to be at most m-dimensional ($m \in \mathbb{N}$) iff for every open cover $\mathcal{B} \subset \mathcal{O}$, there is an open cover $\mathcal{A} \subset \mathcal{O}$ that refines $\mathcal{B}$ and has order at most $m + 1$.*
   The dimension of X is the smallest $m \in \mathbb{N}$ with that property.

Example

Lēi bèi gé

Lebesgue covering dimension (勒贝格 覆盖维度) II

0. $\emptyset$ has the dimension $-\infty$;
1. A discrete space X has topological dimension 0, because the pointwise open cover has only itself as refinement and every point lies in exactly 1 open set.
2. The interval $[0, 1]$ has topological dimension 1, because we can cover it with open sets, such that no 3 of them intersect. Also let $[0, 1)$ and $(0, 1]$ be a cover $\mathcal{B}$, if $\mathcal{A}$ is any refinement of that, we show that $\mathcal{A}$ has order at least 2. Since $\mathcal{A}$ is a refinement it has at least 2 elements. Let $U \in \mathcal{A}$ be one of them and V the union of all the other elements. If $\mathcal{A}$ had order 1, then U and V must be disjoint open sets covering $[0, 1]$. But the interval is connected so this is not possible. Therefore $[0, 1]$ has topological dimension 1.

Properties of topological dimension

Theorem

Given a finite dimensional topological space X.

1. *Every emedded subspace $Y \subset X$ has dimension* $\dim Y \leq \dim X$.
2. *Given $X = Y \cup Z$ of closed subspaces, then* $\dim X = \max\{\dim Y, \dim Z\}$.
3. *If X is compact and metrizable, then it can be embedded into* $\mathbb{R}^{2\dim X+1}$.

Remark

Given finite dimensional topological spaces X and Y, then $\dim X \times Y \geq \dim X + \dim Y$. It can be strictly greater.