Euler's polyhedron formula and Homology theory – 欧拉多面体公式与同调理论

Tóngdiào lǐlùn

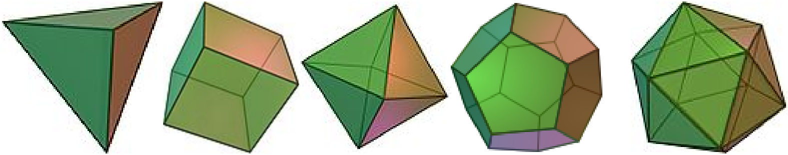
Melchior Grützmann / 古梅西
melchiorG@gMail.com

June 7, 2012

Outline

Regular polyhedra – 正多面体

Consider the regular polyhedra



name	vectices •	edges –	faces Δ	$v - e + f$
tetrahedron	4	6	4	2
cube	8	12	6	2
octahedron	6	12	8	2
dodecahedron	20	30	12	2
icosahedron	12	30	20	2

Theorem (Euler)

Given a convex polyhedron with v vertices, e edges, and f faces, then $v - e + f = 2$.

Proof: Polyhedra tale. A company of soldiers set out to a foreign planet. The surface of the planet was covered with a web of swimming pools, two adjacent swimming pools separated by a wall. One swimming pool was filled with water. The soldiers distribute each on an intersection point of walls. They decide to flood the whole planet by busting some walls. They will bust one wall if there is water on one side, but not on the other side. After the whole planet is flooded, one of them – the commander – blows a whistle and all the other soldiers start running towards him along the walls. When the soldiers are in the middle of their first wall, the commander blows again and the soldiers stop, exactly one in the middle of each wall plus the commander.

Euler's polyhedra formula II

The commander counts the number of walls as follows: Let f be the number of swimming pools, then we blasted exactly $f - 1$ walls, because the first pool was already flooded and with each blasted wall we filled exactly one more pool. When I blew the wistle first, there was from every soldier (except me) exactly one path to me, because two distinct paths would make a loop that enclosed swimming-pools that were not yet flooded. Therefore out of the two soldiers at one wall exactly one of them entered the wall. Let v be the number of soldiers (including me), then the total number of walls is $e = f - 1 + v - 1$ and thus $f - e + v = 2$ as required. □

Generalization

- All the regular polyhedra are inscribed on the sphere $\mathbb{S}^2$, so maybe the 2 is a property of the sphere.
- What happens for non-convex (非凸) polyhedra?

Simplicial complex – 单纯复形

dānchún fùxíng

Definition

1. *A d-dimensional simplex (单纯形) is a set $\{c_0, c_1, \ldots, c_d\}$ of $d+1$ elements. A geometric realization is the convex hull*

$$|\{c_0, \ldots, c_d\}| := \left\{ \sum_{i=0}^{d} t_i c_i : 0 \leq t_i \wedge \sum_i t_i \leq 1 \right\} \qquad (1)$$

   for $c_i \in \mathbb{R}^N$ points in generic position with $N \geq d$. Generic position means that the points are not in an affine subspace of dimension $d-1$.

2. *A simplicial complex S is a finite set of simplices such that for every $\sigma \in S$ and $\sigma' \subset \sigma$ also $\sigma' \in S$. We call $\sigma' \subset \sigma$ the subsimplices of σ and for $\dim \sigma = d$ every $\sigma' \subset \sigma$ with $\dim \sigma' = d-1$ a face of σ.*

   A geometric realization of S means a geometric realization of all simplices in one space $\mathbb{R}^N$ such that $|\sigma| \cap |\sigma'| = |\sigma \cap \sigma'|$ for all $\sigma, \sigma' \in S$.

Examples of simplicial complexes

The polyhedra can be made into examples of geometric realizations of simplicial complexes if we partition the faces into triangles.

1. the tetrahedron $T = \{\{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{1,2\}, \{2,0\}, \{1,3\}, \{3,0\}, \{2,3\}, \{0,1,2\}, \{0,1,3\}, \{0,2,3\}, \{1,2,3\}\}$ – 4 vertices, 6 edges, 4 faces
2. the cube: triangulated with 8 vertices, 18 edges, 12 triangles;
3. the octahedron with 6 vertices, 12 edges, 8 triangles;
4. the dodecahedron triangulated with 20 vertices, $30 + 2 \cdot 12 = 54$ edges, and $3 \cdot 12 = 36$ triangles;
5. the icosahedron with 12 vertices, 30 edges, 20 triangles;

They are all triangulations of the sphere.

Orientation – 定向 (dìngxiàng)

Definition

Given a simplex $\sigma = \{0, 1, \ldots, d\}$ *we say that two permutations* $s = (s_0, s_1, \ldots, s_d)$ *and* $t = (t_0, t_1, \ldots, t_d)$ *of* σ *have the same orientation iff we can transform s into t by an even number of flips. We denote the equivalence class of* $(s_0, s_1, \ldots, s_d)$ *as* $[s_0, s_1, \ldots, s_d]$.

Example

Given the simplex $\{0, 1, \ldots, d\}$, then $[0, 1, \ldots, d]$ is one orientation and $[1, 0, 2, 3, \ldots, d]$ is the other orientation.

An oriented simplicial complex S_+ is a simplicial complex S together with the choice of an orientation for every simplex $\sigma \in S$.

Simplicial chains

Definition

Given a simplicial complex S with orientation S_+, then the space of simplicial chains is the free abelian group generated by the positively oriented simplices, i.e. $C_\bullet(S) := \mathbb{Z}[S_+]$*. We say that a chain* $c := \sum_i c_i\sigma_i$ *with* $c_i \in \mathbb{Z}$ *and* $\sigma_i \in S$ *has degree d iff for every* $c_i \neq 0$*, the corresponding* σ_i *has dimension d. Then we write* $c \in C_d(S)$.

Example

Given the 3d simplex T, then $C_0(T) = \langle [0],[1],[2],[3] \rangle_\mathbb{Z}$; the chains of degree 1, $C_1(T) = \langle [0,1],[0,2],[0,3],[1,2],[1,3],[2,3] \rangle_\mathbb{Z}$, and the chains of degree 2,
$C_2(T) = \langle [0,1,2],[0,1,3],[0,2,3],[1,2,3] \rangle_\mathbb{Z}$.
We use the convention $[1,0,2] = -[0,1,2]$.

Chain complex – 链 (liàn) 复形 I

Definition

Given a simplicial complex S, then the boundary operator is $\partial\colon C_p(S) \to C_{p-1}(S)$ *with*

$$\partial[c_0, \ldots, c_p] = \sum_{i=0}^{p} (-1)^i [c_0, \ldots, \hat{c}_i, \ldots, c_p] \tag{2}$$

and linearly extended to all chains.

Example

Starting from the tetrahedron T, the boundary operator is $\partial[0,1,2] = [1,2] - [0,2] + [0,1]$ and correspondingly for the other simplices in $C_2(T)$, $\partial[0,1] = [1] - [0]$, ..., and $\partial[0] = 0 = \partial[1] = \partial[2] = \partial[3]$. Note that $\partial^2[0,1,2] = 0$.

Chain complex – 链(liàn)复形 I

Definition

Given a simplicial complex S, then the boundary operator is $\partial\colon C_p(S) \to C_{p-1}(S)$ *with*

$$\partial[c_0, \ldots, c_p] = \sum_{i=0}^{p} (-1)^i [c_0, \ldots, \hat{c}_i, \ldots, c_p] \qquad (2)$$

and linearly extended to all chains.

Example

Starting from the tetrahedron T, the boundary operator is $\partial[0,1,2] = [1,2] - [0,2] + [0,1]$ and correspondingly for the other simplices in $C_2(T)$, $\partial[0,1] = [1] - [0]$, ..., and $\partial[0] = 0 = \partial[1] = \partial[2] = \partial[3]$. Note that $\partial^2[0,1,2] = 0$.

Proposition

$\partial^2 = 0$ *for every simplicial complex.*

Therefore $(C_\bullet(S), \partial)$ is called the chain complex of S.

Chain complex – 链 (liàn) 复形 II

Proof.

Let thus $S \ni \sigma = [c_0, \ldots, c_p]$, then

$$\partial\sigma = \sum_{i=0}^{p} (-1)^i [c_0, \ldots, \hat{c}_i, \ldots, c_p],$$

$$\begin{aligned}
\partial^2\sigma &= \sum_{i=0}^{p} \Bigg(\sum_{j=0}^{i-1} (-1)^{i+j} [c_0, \ldots, \hat{c}_j, \ldots, \hat{c}_i, \ldots, c_p] + \\
&\quad + \sum_{j=i+1}^{p} (-1)^{i+j-1} [c_0, \ldots, \hat{c}_i, \ldots, \hat{c}_j, \ldots, c_p] \Bigg) \\
&= \sum_{i<j} (-1)^{i+j} \big([c_0, \ldots, \hat{c}_i, \ldots, \hat{c}_j, \ldots, c_p] - \\
&\quad - [c_0, \ldots, \hat{c}_i, \ldots, \hat{c}_j, \ldots, c_p] \big) \\
&= 0.
\end{aligned}$$

Homology – 同调 (diào)

Definition

Given a complex $(C_\bullet, \partial)$, *then its boundaries are the chains* $B_\bullet := \operatorname{im}\partial$, *its cycles are the chains* $Z_\bullet := \ker\partial$ *and its homology is* $H_p(C_\bullet, \partial) := Z_p/B_p$.

Example

Going back to the tetrahedron, we obtain the cycles:
$Z_0 = \langle [0], [1], [2], [3] \rangle_{\mathbb{Z}}$,
$Z_1 = \langle [0,1]+[1,2]+[2,0], [0,1]+[1,3]+[3,0], [0,2]+[2,3]+[3,0] \rangle_{\mathbb{Z}}$,
$Z_2 = \langle [0,2,1] + [0,1,3] + [1,2,3] + [0,3,2] \rangle_{\mathbb{Z}}$.
The boundaries are $B_0 = \langle [1]-[0], [2]-[1], [3]-[2] \rangle_{\mathbb{Z}} \subset Z_0$, $B_1 = Z_1$, and $B_2 = 0 \subset Z_2$.
Therefore the homology of the tetrahedron is $H_0(T) \cong \mathbb{Z}$, $H_1(T) = 0$, $H_2(T) \cong \mathbb{Z}$.

Independence of triangulation

Theorem

Given a surface, then its simplicial homology is independent of the triangulation.

This is easy to see for a refinement, but for arbitrary triangulations it requires singular homology a generalization of simplicial homology.

Euler characteristic – 欧拉示(shì)性数

Theorem

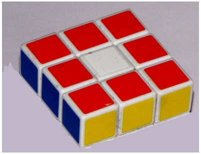
Given a surface S, then its Euler characteristic $\chi(S) := \sum_{d\geq 0}(-1)^d \dim C_d(S)$ computes as

$$\chi(S) = \sum_{d\geq 0}(-1)^d \dim H_d(S) \tag{3}$$

for any triangulation of S.

For the tetrahedron we obtain $\chi(T) = 1 - 0 + 1 = 2$ coinciding with the initial results. But also for the other regular simplices we obtain the same result, because they all refine to triangulations of the sphere.
To obtain a different result, we need to drop the convexity condition, i.e. triangulate a different surface.

Torus – 环面



Partitioned into 8 cubes, thus having $8 \cdot 4 = 32$ vertices, $4 \cdot 16 = 64$ edges, $4 \cdot 8 = 32$ faces and therefore the Euler characteristic $\chi(\mathbb{T}) = v - e + f = 0$.

Literature

- A. Hatcher: *Algebraic Topology*, 2nd edn., 556 pp. (Cambridge University Press, 2002), ISBN 978-0521795401. www.math.cornell.edu/~hatcher.
- J. Munkres: *Topology*, 2nd edn., 537 pp. (Prentice Hall, 2000), ISBN 0131816292.
- written by the web: *Wikipedia the free encyclopedia*. http://www.wikipedia.org/.