Lie and Courant algebroids

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1 Introduction

Lie algebras play an important role in geometry as they describe the infinitesimal symmetries of space as well as of objects in space. A Lie algebra consists of a (usually finite dimensional) vector space $\mathfrak{g}$ together with a skew-symmetric bilinear operation $[.,.]$, that however is not associative, but fulfills the Jacobi identity (1).

The simplest non-trivial example is $\mathfrak{so}(3)$, the vector space $\mathbb{R}^3$ together with the vector product $\times$.

Lie algebras permit an extensive study such as classification, representation theory, or cohomology theory. Some of them have additional structure, namely a non-degenerate symmetric bilinear inner product $\langle .,. \rangle$ that is compatible in the following sense of (5).

The inner product simplifies the study of the Lie algebra. They are called quadratic Lie algebras. For semi-simple Lie algebras there is a standard construction due to Killing to obtain an inner product.

The research area of algebroids wants to generalize the notion of Lie algebra to a vector bundle $A \to M$ over a smooth manifold M. The simplest example is the Lie algebroid which beside a skew-symmetric bracket on the sections $\Gamma(A)$ also has an anchor map $\rho\colon A \to TM$ subject to a Leibniz rule (2).

Another example are Courant algebroids which generalize the notion of quadratic Lie algebra.

2 Definitions

Definition 1. *A* Lie algebroid *is a vector bundle $A \to TM$ together with a skew-symmetric bracket $[.,.]$ on its sections and a vector bundle morphism $\rho\colon A \to TM$ called the anchor, subject to the rules*

$$[\phi, [\psi, \chi]] = [[\phi, \psi], \chi] + [\psi, [\phi, \chi]] \tag{1}$$
$$[\phi, f \cdot \psi] = \rho(\phi)[f] \cdot \psi + f \cdot [\phi, \psi] \tag{2}$$

where $\phi, \psi, \chi \in \Gamma(A)$.

Example 2. 0. Lie algebras

1. the tangent bundle
2. Given a vector bundle $V \to M$, then its frame bundle $\mathcal{F}(V)$ is a principal bundle with structure group $G = \mathrm{GL}_k(\mathbb{R})$ where $\mathrm{rk}\, V = k$. Then $T\mathcal{F}(V)/G$ is a Lie algebroid over M that acts on V.

Definition 3. *A* Courant algebroid *is a vector bundle $E \to M$ together with three operations, a bilinear bracket $[.,.]\colon \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$, a fiberwise non-degenerate symmetric bilinear form $\langle .,. \rangle\colon E \times E \to M \times \mathbb{R}$, and a morphism of vector bundles $\rho\colon E \to TM$ called the anchor map, subject to the rules*

$$[\phi, [\psi, \chi]] = [[\phi, \psi, \chi] + [\psi, [\phi, \chi]] \tag{3}$$
$$[\phi, f \cdot \psi] = \rho(\phi)[f] \cdot \psi + f \cdot [\phi, \psi] \tag{4}$$
$$\rho(\phi)\langle \psi, \psi \rangle = 2\langle [\phi, \psi], \psi \rangle \tag{5}$$
$$[\phi, \phi] = \tfrac{1}{2}\rho^* \mathrm{d}\langle \phi, \phi \rangle \tag{6}$$

where $\phi, \psi, \chi \in \Gamma(E)$ and $f \in \mathrm{C}^\infty(M)$.

Example 4. 0. A quadratic Lie algebra $(\mathfrak{g}, [.,.], \langle .,. \rangle)$ is a Courant algebroid over a point.

1. $E := TM \oplus T^*M$ with the inner product $\langle X \oplus \alpha, Y \oplus \beta \rangle := \alpha(Y) + \beta(X)$, anchor $\rho(X \oplus \alpha) := X$ and for $H \in \Omega^3(M)$ with $\mathrm{d}H = 0$ we define the Ševera bracket as

$$[X \oplus \alpha, Y \oplus \beta]_H := [X, Y] \oplus \mathcal{L}_X \beta - i_Y \mathrm{d}\alpha + H(X, Y, \cdot).$$

   Then this is a Courant algebroid.

2. (Double of a Lie bialgebroid, Liu–Weinstein–Xu)[LWX97] Given a Lie algebroid $(A, [.,.]_A, \rho_A)$ together with the structure of a Lie algebroid on its dual vector bundle $(A^*, \rho_*, [.,.]_*)$, the latter introduces a Lie algebroid differential $\mathrm{d}_*\colon \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+1} A)$. This forms a Lie bialgebroid iff

$$\mathrm{d}_*[\phi, \psi] = [\mathrm{d}_*\phi, \psi]_A + [\phi, \mathrm{d}_*\psi]_A$$

   for all $\phi, \psi \in \Gamma(A)$ where the bracket on A is extended to multisections using a graded Leibniz rule. Then $E := A \oplus A^*$ is endowed with the structure of a Courant algebroid, namely $\langle \phi \oplus \alpha, \psi \oplus \beta \rangle := \alpha(\psi) + \beta(\phi)$, $\rho(\phi \oplus \alpha) = \rho_A(\phi) + \rho_*(\alpha)$, and

$$[\phi \oplus \alpha, \psi \oplus \beta] = [\phi, \psi]_A + \mathcal{L}^*_\alpha \psi - i_\beta \mathrm{d}_* \psi \oplus [\alpha, \beta]_* + \mathcal{L}^A_\phi \beta - i_\psi \mathrm{d}_A \beta$$

   where $\alpha, \beta \in \Gamma(A^*)$.

3 Realization as Q-manifolds

Definition 5 (N-manifold)**.** *An* N-manifold *is a ringed Hausdorff second countable space $(M, \mathcal{O})$ with a sheaf of unital associative graded commutative algebras, that are locally free, i.e. there is a finite sequence $p_\bullet = (p_0, p_1, \ldots, p_N)$ of non-negative integers such that locally*

$$\mathcal{O}(U) \cong \mathrm{C}^\infty(\tilde{U}) \otimes \wedge^\bullet \mathbb{R}^{p_1} \otimes S^\bullet \mathbb{R}^{p_2} \otimes \ldots \mathbb{R}^{p_N}.$$

$\dim(M, \mathcal{O}) := p_\bullet$.

Example 6. 0. an ordinary smooth manifold M is a trivial N-manifold.

1. Given an N-graded vector bundle $E \to M$, then M with the structure sheaf $\mathcal{O}(U) := \Gamma(S^\bullet E^*)$ where $S^\bullet E^*_{2k}[2k]$ is the symmetric algebra and $S^\bullet E^*_{2k+1}[2k+1]$ is the skew-symmetric algebra, then this is an N-manifold.

N-manifolds permit the construction of tangent and cotangent bundles. The latter is generally only Z-graded, but can be made N-graded again by shifting the fiber-degrees accordingly. The exterior algebra $\wedge^\bullet T^*\mathcal{M}$ of an N-graded manifold is double graded, i.e. forms have a form degree, which for $\alpha \in \wedge^p T^*\mathcal{M}$ is p, and a coordinate degree coinciding with the function degree for exact 1-forms $\mathrm{d}f$, $f \in \mathcal{O}(\mathcal{M})$.

The notions of Poisson and symplectic manifolds generalize to N-manifolds straight-forwardly. We require the structures to be of homogeneous degree.

Example 7 (Roytenberg)**.** Given a pseudo Euclidean vector bundle $(E, \langle .,. \rangle)$, we can interpret $E[1]$ as a Poisson manifold with odd fibers and Poisson bracket of degree -2 via

$$\{\phi, \psi\} := \langle \phi, \psi \rangle, \quad \{\phi, f\} = 0 = \{f, g\}$$

for $\phi, \psi \in \Gamma(E^*) \cong \Gamma(E)$, $f, g \in \mathrm{C}^\infty(M)$ and extended using the Leibniz rule.

Its minimal symplectic realization is a graded symplectic manifold $(\mathcal{E}, \omega)$ together with a surjective morphism of Poisson manifolds $p\colon \mathcal{E} \to E[1]$.

Definition 8. *A Q-structure on a graded manifold $\mathcal{M}$ is a vector field of degree 1, that commutes with itself, i.e. $Q \in \Gamma(T\mathcal{M})$, $[Q, Q] = 0$.*

Example 9. compatible Q-structures on symplectic N-manifolds are Hamiltonian. A symplectic manifold of degree 1 is an odd cotangent bundle $\mathcal{M} = T^*[1]M$. A Q-structure on this is generated by $H = \Pi \in \Gamma(\wedge^2 TM)$ which has to be Poisson, i.e. $0 = \{H, H\} = [\Pi, \Pi]$.

Theorem 10 (Roytenberg)**.** *Given a pseudo-Euclidian vector bundle E, then Courant structures on E correspond 1:1 with Q-structures $\Theta \in \mathcal{O}_{[3]}(\mathcal{E})$ on the minimal symplectic realization $\mathcal{E}$ via:*

$$\rho(\phi)[f] = \{\{\Theta, \phi\}, f\}, \tag{7}$$
$$[\phi, \psi] = \{\{\Theta, \phi\}, \psi\}, \tag{8}$$
$$\langle \phi, \psi \rangle = \{\phi, \psi\} \tag{9}$$

where $\phi, \psi \in \Gamma(E) \cong \Gamma(E^*) = \mathcal{O}_{[1]}(\mathcal{E})$ and $f \in \mathrm{C}^\infty(M) = \mathcal{O}_{[0]}(\mathcal{E})$.

We can thus define the standard cohomology of a Courant algebroid E as $H^\bullet_{std}(E) := H^\bullet(\mathcal{O}(\mathcal{E}), Q)$.

4 Matched pairs

The notion of matched pairs tries to capture two algebroids with additional structure such that their direct sum is again an algebroid.

Definition 11 (Mokri)**.** *A matched pair of Lie algebroids $(A_i, \rho_i, [.,.]_i)$ with $i = 1, 2$ is given by a flat connection of each algebroid on the other vector bundle, i.e. $\overrightarrow{\nabla}\colon \Gamma(A_1) \otimes \Gamma(A_2) \to \Gamma(A_2)$ and $\overleftarrow{\nabla}\colon \Gamma(A_2) \otimes \Gamma(A_1) \to \Gamma(A_1)$ subject to the rules*

$$\overrightarrow{\nabla}_\phi [\alpha, \beta]_2 = [\overrightarrow{\nabla}_\phi \alpha, \beta]_2 + [\alpha, \overrightarrow{\nabla}_\phi \beta]_2 + \overrightarrow{\nabla}_{\overleftarrow{\nabla}_\beta \phi} \alpha - \overrightarrow{\nabla}_{\overleftarrow{\nabla}_\alpha \phi} \beta \tag{10}$$
$$\overleftarrow{\nabla}_\alpha [\phi, \psi]_1 = [\overleftarrow{\nabla}_\alpha \phi, \psi]_1 + [\phi, \overleftarrow{\nabla}_\alpha \psi]_1 + \overleftarrow{\nabla}_{\overrightarrow{\nabla}_\psi \alpha} \phi - \overleftarrow{\nabla}_{\overrightarrow{\nabla}_\phi \alpha} \psi \tag{11}$$

Example 12. 0. Given two Lie algebras with mutual representations, then their twisted sum is again a Lie algebra and therefore they form a matched pair of Lie algebras.

1. Given a Lie algebroid A together with a flat A-connection on a vector bundle V, then $A \oplus V$ can be endowed with a Lie algebroid structure (the bracket on V is trivial). This is a matched pair of Lie algebroids.

Theorem 13 (Mokri)**.** *Given a matched pair of Lie algebroids (A_1, A_2), then their direct sum is endowed with the structure of a Lie algebroid and the A_i are Lie subalgebroids.*

This becomes obvious in the graded picture where the Q-structure of the Lie algebroid is the Lie algebroid differential.

Definition 14 (MPCA, G&Stiénon)**.** *A matched pair of Courant algebroids are two Courant algebroids $(E_i, \langle .,. \rangle_i, \rho_i, [.,.]_i)$, $i = 1, 2$ over the same base M together with an inner product preserving connection of each algebroid on the vector bundle of the other algebroid, such that their direct sum is again a Courant algebroid.*

Theorem 15. *Given two Courant algebroids $(E_i, \langle .,. \rangle_i, \rho_i, [.,.]_i)$, $i = 1, 2$ together with mutual inner product preserving connections $\overleftarrow{\nabla}$ and $\overrightarrow{\nabla}$, then they form a matched pair of Courant algebroids iff they are subject to the 5 structure equations ….*

The bracket on the sum reads:

$$\begin{aligned}[\phi \oplus \alpha, \psi \oplus \beta] = {} & [\phi, \psi]_1 + \overleftarrow{\nabla}_\alpha \psi - \overrightarrow{\nabla}_\beta \phi + \langle \overrightarrow{\nabla} \alpha, \beta \rangle_2 \\ & \oplus [\alpha, \beta]_2 + \overrightarrow{\nabla}_\phi \beta - \overrightarrow{\nabla}_\psi \alpha + \langle \overleftarrow{\nabla} \phi, \psi \rangle_1 \end{aligned} \tag{12}$$

where again $\phi, \psi \in \Gamma(E_1)$ and $\alpha, \beta \in \Gamma(E_2)$. Remember that the inner product is

$$\langle \phi \oplus \alpha, \psi \oplus \beta \rangle = \langle \phi, \psi \rangle_1 + \langle \alpha, \beta \rangle_2.$$

Matched pairs of Courant algebroids also have a nice description in the language of supermanifolds, namely they are the sum of two compatible Q-structures of the fibered coproduct of the separate minimal symplectic realizations.

Example 16. 0. Given two quadratic Lie algebras with mutual orthogonal representations, then their sum vector space can be endowed with the structure of a quadratic Lie algebra again. They form therefore a matched pair of quadratic Lie algebras.

1. (Merker) Given a Courant algebroid $(E, \langle .,. \rangle_1, \rho_1, [.,.]_1)$ together with a flat connection on a pseudo-Euclidean vector bundle $(V, \langle .,. \rangle_2)$ over the same base M that preserves the inner product. Then $E \oplus V$ can be endowed with the structure of a Courant algebroid. This type of matched pairs play an important role in the theory of Hamiltonian systems with ports.

4.1 Structure of regular Courant algebroids

Regular means that $\ker \rho \subset E$ is a vector bundle, i.e. has constant rank.

Theorem 17 (Chen–Stiénon–Xu)**.**

Corollary 18 (G&Stiénon)**.** *Given a flat regular Courant algebroid, then this is a matched pair of a generalized standard Courant algebroid $E_1 := (F \oplus F^*)_H$ with $F \subset TM$ integrable and a bundle of quadratic Lie algebras $E_2 := \mathfrak{g}$.*

5 Morphisms and coisotropic calculus

Remark 19. Coisotropic calculus was introduced by A. Weinstein in [Wei88]. A coisotropic submanifold of the product $\bar{M} \times N$ of two Poisson manifolds M and N is a generalization of morphism of Poisson manifolds, namely $L := \operatorname{graph} \phi$ for $\phi\colon M \to N$ is a coisotropic submanifold iff ϕ is a Poisson morphism. We wish to generalize this notion to Lie and Courant algebroids.

Together with Zh. Chen we want to study morphisms, weak, and generalized morphisms of Courant algebroids and develop conditions analogously to those of Weinstein.

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