Abstract algebra: Homework 7 – Solutions

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1.13 Normal series and Jordan–Hölder Theorem

Exercise 1.13.1. a. Show that D_4 has a normal series where one of the components is not a normal subgroup of D_4.

b. Given normal series for $N \triangleleft G$ and G/N. Show that these can be pieced together to give a normal series of the group G.

c. Let $\{\mathrm{id}\} \triangleleft G_1 \triangleleft G_2 \triangleleft \ldots \triangleleft G_n = G$ be a normal series. Explain how normal series of each factor G_k/G_{k-1} give a refinement of this normal series. What is needed to obtain a composition series?

d. Given composition series of $N \triangleleft G$ and G/N. How can you obtain a composition series for the group G?

Solution.

a. The normal subgroups of S_4 are $\{\mathrm{id}\}, A_4, S_4$ thus the normal series $\{\mathrm{id}\} \triangleleft \langle(12)(34)\rangle \triangleleft \langle(12)(34), (13)(24), (14)(23)\rangle \triangleleft A_4 \triangleleft S_4$ and $\langle(12)(34)\rangle$ is not normal in S_4.

c. Given any normal series $\{1\} \triangleleft H_1 \triangleleft H_2 \triangleleft \ldots \triangleleft G_n/G_{n-1}$, then the inverse of the projection map $\pi\colon G_n \twoheadrightarrow G_n/G_{n-1}$ produces a normal series $G_{n-1} \triangleleft \pi^{-1}H_1 \triangleleft \pi^{-1}H_2 \triangleleft \ldots \triangleleft G_n$ that fits into the given normal series of G. If we want a composition series of G, then we need composition series for each factor, because H_k/H_{k-1} is simple iff $\pi^{-1}H_k/\pi^{-1}H_{k-1}$ is simple.

d. That is a special case of part 1.13.1c with a short normal series $\{1\} \triangleleft N \triangleleft G$.

□

Exercise 1.13.2. If G has a composition series, then every normal subgroup $N \triangleleft G$ and every quotient G/N (by a normal subgroup) has a composition series.
Hint: Show how N appears in a composition series.

Solution. Remember that given the existence of one composition series, then every normal series can be refined to a composition series. Thus starting from the normal series $\{\mathrm{id}\} = G_0 \triangleleft N \triangleleft G$, then we can refine it to a composition series $\{\mathrm{id}\} \triangleleft G_1 \triangleleft G_2 \triangleleft \ldots \triangleleft G_n = N \triangleleft G_{n+1} \triangleleft \ldots \triangleleft G_N = G$. But then $(G_k)_{k=0}^{n}$ is a composition series of N, as well as $\pi\colon G \twoheadrightarrow G/N$ reduces $(G_k)_{k=n}^{N}$ to a composition series of G/N. □

Exercise 1.13.3. Find all composition series of

a. A_4,

b. D_4,

c. D_5.

Solution.

a. Remember the composition series $\{\mathrm{id}\} = G_0 \triangleleft G_1 = \langle(12)(34)\rangle \triangleleft G_2 = \langle(12)(34),(13)(24),(14)(23)\rangle \triangleleft$ A_4 with the factors $F_1 := G_1/G_0 = G_1 \cong C_2$, $F_2 := G_2/G_1 \cong C_2$, and $F_3 := A_4/G_2 \cong C_3$. Thus all composition series have the factors $\{C_2, C_2, C_3\}$ and length 3. We thus first look for normal subgroups of index 2 or 3, but there is only one $V_4 = G_2$, so the last factor has to be C_3. Beside the given G_1 there are 2 more subgroups of V_4 of index 2, they are $G_{1,2} = \langle(13)(24)\rangle$ and $G_{1,3} = \langle(14)(23)\rangle$. So there are exactly 3 composition series of A_4.

b. Remember that $D_4 = \langle\sigma, \tau : \sigma^2 = \mathrm{id} = \tau^4, \sigma\tau\sigma = \tau^{-1}\rangle$ has 3 non-trivial normal subgroups $G_{2,1} = \langle\tau\rangle$, $G_{2,2} = \langle\sigma, \tau^2\rangle \cong D_2$ and $G_{1,1} := \operatorname{cent} D_4 = \langle\tau^2\rangle \subset G_{2,k}$. So the composition series of D_4 are $\{\mathrm{id}\} = G_0 \triangleleft G_{1,1} \triangleleft G_{2,1} \triangleleft D_4$, $\{\mathrm{id}\} = G_0 \triangleleft G_{1,1} \triangleleft G_{2,2} \triangleleft D_4$, and $\{\mathrm{id}\} \triangleleft \langle\sigma\rangle \triangleleft G_{2,2} \triangleleft D_4$. All with three factors $F_k \cong C_2$.

c. For $D_5 = \langle\sigma, \tau : \sigma^2 = \mathrm{id} = \tau^5, \sigma\tau\sigma = \tau^{-1}\rangle$ it is even smaller, because $G_1 := \langle\tau\rangle$ is the only nontrivial subgroup and it is simple by itself. So the composition series is $\{\mathrm{id}\} \triangleleft G_1 \triangleleft D_5$ (with composition factor $D_5/G_1 \cong C_2$).

$\square$

Exercise 1.13.4. Show that all abelian groups of order n have the same simple factors.

Solution. Let $n = p_1^{n_1} \cdots p_k^{n_k}$ for different primes $p_i \in \mathbb{P}$ and positive integers $n_i \in \mathbb{N}^*$. We know that every abelian group A of order n is the direct sum of abelian p_i-groups of order $p_i^{n_i}$. It is thus sufficient to show that every abelian p-group of order p^n with $n \geq 1$ has the same composition factors. As a conclusion of Sylow's theorem, we know that each of them has an element of order p. Therefore the first term of a composition series is $\{0\} \triangleleft C_p \triangleleft \ldots \triangleleft A$ with the first factor being C_p itself. By induction over n we thus see that all abelian p-groups of order p^n have exactly n factors C_p in every of their composition series. $\square$

Exercise 1.13.5. Show that the simple factors of D_n are all abelian. (This means that D_n is solvable.)

Solution. The two cases of Exercise 1.13.3bc show what can happen with D_n. We know that they all have a normal subgroup $N := \langle\tau\rangle$ of index 2, so one of their factors is C_2 which is abelian. Already in the short normal series $\{\mathrm{id}\} \triangleleft N \triangleleft G$ the other factor is N itself which is cyclic and thus abelian. Even if we refine N we end up with possibly more but all abelian factors. □

Exercise 1.13.6. Let G be a group of order n and m the length of each of its composition series.

a. Show that a group of order $n = p^m$ where $p \in \mathbb{P}$ is a prime and $m \in \mathbb{N}_+$ a positive integer has a composition series of length m.

b. Show that $m \leq \log_2 n$.

c. Show that equality $m = \log_2 n$ is possible for arbitrary high values of n.

Solution.

a. By Proposition 1.14.9 G has a normal subgroup of order p^{n-1} with factor C_p and thus by induction G has a normal series with factors C_p each and of length m.

b. Let $n = p_1^{n_1} \cdots p_k^{n_k}$ with distinct primes $p_i \in \mathbb{P}$ thus $p_i \geq 2$ and positive exponents $n_i \in \mathbb{N}^*$ which thus add up $n = \prod_i p_i^{n_i} \geq \prod_i 2^{n_i} \geq 2^m$ where the last step is since the group can have at most $\sum_i n_i$ composition factors, namely if it is solvable, and thus $m \leq \log_2 n$.

c. Every 2-group of order 2^m has composition series all of length m for arbitrarily large $m \in \mathbb{N}$.

□

Exercise 1.13.7. What can you say if $DG = G$ for a group G?
Hint: Consider $D(A_5 \times A_5)$ and note that this direct product is not simple.

Solution. The group is semi-simple, i.e. meaning the (semi)-direct product of simple non-solvable groups. The theory of semi-simple groups actually shows that these products are all direct products. □

1.14 Nilpotent groups

Exercise 1.14.1. Prove the proposition about the upper central series, i.e. a group is nilpotent iff its upper central series $Z_{n+1} := \{z \in G : \forall g \in G : [g, z] \in Z_n\}$ with $Z_0 := \{\mathrm{id}\}$ ends in G.
Hint: Suppose G is nilpotent of length n (i.e. $G_n = 1$), show that $Z_k \supset G_{n-k}$ and thus $Z_n = G_0 = G$. In the other direction show that $Z_n = G$ implies $G_k \subset Z_{n-k}$.

Solution. "⇒" Consider ...
"⇐" Consider conversely ... □

1.15 Group extensions & Semi-direct products

Exercise 1.15.1. Find all group extensions in the following cases

a. of $Q = C_2$ by $N = C_3$,

b. of $Q = C_3$ by $N = C_2$.

c. Which of the extensions $1 \to C_3 \to G \to C_2 \to 1$ are semi-direct products?

Solution.

a. We see that $|G| = |Q|\,|N| = 6$ and there are only two groups of order 6 C_6 and $S_3 \cong D_3$ each of which has a unique normal subgroup of order 3. So $1 \to C_3 \to C_6 \to C_2 \to 0$ as well as $1 \to C_3 \to D_3 \to C_2 \to 0$ are the only two extensions.

b. D_3 does not have a normal subgroup of order 2 and C_6 has a unique one, so $1 \to C_2 \to C_6 \to C_3 \to 0$ is the only extension in that way.

c. Remember that $C_6 \cong C_2 \times C_3$ so this is a direct product, i.e. a trivial semi-direct product. Conversely $e\colon C_3 \to D_3 : 1 \mapsto \sigma$ with $\rho\colon C_2 \to \mathrm{Aut}(C_3) : 1 \mapsto (\tau \mapsto \tau^{-1})$ exposes $D_3 = C_2 \,{}_\rho\!\ltimes C_3$ as a (non-trivial) sermi-direct product.

□