Abstract algebra: Homework 6 – Solutions

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1.10 Small groups

Exercise 1.10.1. To which group of order 12 in Table 1.10.1 is $C_2 \oplus D_3$ isomorphic?

Solution. As we have shown earlier cent $D_6 = \langle \tau^3 \rangle \cong C_2$ and the other two non-trivial normal subgroups are $C_6 \cong \langle \tau \rangle \triangleleft D_6$ and $D_3 \cong \langle \sigma, \tau^2 \rangle \triangleleft D_6$ with $C_2 \cap D_3 = \langle \emptyset \rangle$ so $D_6 \cong C_2 \oplus D_3$. □

Non-abelian groups in the following exercise should be specified by (easy) presentations.

Exercise 1.10.2. Find all groups of order
a. 51, **b.** 21, **c.** 39, **d.** 55, **e.** 57, **f.** 93.

Solution.

a. $51 = 3 \cdot 17$ so $n_3 \equiv 1 \pmod 3$ and $n_3 | 17$, i.e. $n_3 = 1 = n_{17}$ and thus $G = P_3 \times P_{17} \cong \mathbb{Z}/(3) \times \mathbb{Z}/(17) \cong \mathbb{Z}/(51)$, unique, cyclic, abelian.

b. $21 = 3 \cdot 7$, so $n_3 \equiv 1 \pmod 3$ and $n_3 | 7$. Thus either $n_3 = 1$ which leads to $G \cong \mathbb{Z}/(21)$ or $n_3 = 7$ which leads to the short exact sequence $1 \to \mathbb{Z}/(7) \to G \to \mathbb{Z}/(3) \to 0$ where $\mathbb{Z}/(7)$ is the unique normal 7-subgroup of G and $\mathbb{Z}/(3)$ is the image of the quotient map $\pi \colon G \to G/\mathbb{Z}/(7)$. On the other hand we know that there is an embedding $e \colon \mathbb{Z}/(3) \to G$ (actually there are 7 such embeddings) and since $e(\mathbb{Z}/(3)) \cap \mathbb{Z}/(7) = \{\mathrm{id}\} \subset G$, we see that $\pi \circ e = \mathrm{Id}_{\mathbb{Z}/(3)}$, i.e. G is a semi-direct product $G = \mathbb{Z}/(3)_\rho \ltimes \mathbb{Z}/(7)$ (where e is a right-splitting). These semi-direct products are classified by an action $\rho \colon \mathbb{Z}/(3) \to \mathrm{Aut}(\mathbb{Z}/(7))$ since $\mathrm{Aut}(\mathbb{Z}/(7)) \cong (\mathbb{Z}/(7))^* \cong \mathbb{Z}/(6)$ the generators of $\mathbb{Z}/(7)$ and also $3 | \operatorname{ord}_\bullet k$ the choices are $k := \rho(1) \equiv 1, 2, 4 \pmod 7$. $k \equiv 1$ leads to the direct product (with $n_3 = 1$) while $k \equiv 2$ leads to a non-abelian group $\mathbb{Z}/(3)_{\rho_2} \ltimes \mathbb{Z}/(7) = \langle \sigma, \tau : \sigma^3 = \mathrm{id} = \tau^7, \sigma\tau\sigma^{-1} = \tau^2 \rangle$.* Let $G_3 := \mathbb{Z}/(3)_{\rho_4} \ltimes \mathbb{Z}/(7) = \langle \sigma, \tau_4 : \sigma^3 = \mathrm{id} = \tau_4^7, \sigma\tau\sigma^{-1} = \tau_4^4 \rangle$, then we can see that the elements $\sigma\tau^m$ have order 2 while the $\langle \tau \rangle$ have oder 7 or 1, respectively. But $(\sigma\tau^m)\tau^n(\sigma\tau^m)^{-1} = \tau^{2m}$, so there are no elements that fulfill all three relations in G_3, i.e. we have two different solutions.

c. $39 = 3 \cdot 13$ which correspondingly leads to $n_{13} = 1$ and either $n_3 = 1$ with $G \cong \mathbb{Z}/(39)$ or $n_3 = 13$ which leads futher to $G \cong \langle \sigma, \tau : \sigma^3 = \mathrm{id} = \tau^{13}, \sigma\tau\sigma^{-1} = \tau^k \rangle$ with $k \equiv 1, 3, 9 \pmod{13}$. Here $k \equiv 1$ again leads to the direct product (with $n_3 = 1$) while $k \equiv 3$ and $k \equiv 9$ lead to two different non-abelian groups.

*One can see that $\langle \tau \rangle \subset G$ is a normal subgroup of order 7.

d. $55 = 5 \cdot 11$ which correspondingly leads to $n_{11} = 1$ and either $n_5 = 1$ with $G \cong \mathbb{Z}/(55)$ or $n_5 = 11$. The latter ones are $G \cong \langle \sigma, \tau : \sigma^5 = \mathrm{id} = \tau^{11}, \sigma\tau\sigma^{-1} = \tau^k \rangle$ with $k \equiv 1, 4, 5, -2, 3 \pmod{11}$ where $k = 1$ gives the cyclic group ($n_5 = 1$) and $k \equiv 4$ gives a non-abelian extension. **MG:** What about the $k \equiv 5, -2, 3$, are any isomorphic?

e. $57 = 3 \cdot 19$ which leads to $n_{19} = 1$ and either $n_3 = 1$ with $G \cong \mathbb{Z}/(3) \times \mathbb{Z}/(19) \cong \mathbb{Z}/(57)$ or $n_3 = 19$ with $G \cong \langle \sigma, \tau : \sigma^3 = \mathrm{id} = \tau^{19}, \sigma\tau\sigma^{-1} = \tau^k \rangle$ where $k \equiv 1, 7, -8 \pmod{19}$ where $k \equiv 1$ gives the cyclic group ($n_3 = 1$) and $k \equiv 7$ and $k \equiv -8$ give two different non-abelian extensions.

f. $93 = 3 \cdot 31$ which leads correspondingly to $n_{31} = 1$ and either $n_3 = 1$ with $G \cong \mathbb{Z}/(3) \times \mathbb{Z}/(31)$ or $n_3 = 31$ with $G \cong \langle \sigma, \tau : \sigma^3 = \mathrm{id} = \tau^{31}, \sigma\tau\sigma^{-1} = \tau^k \rangle$ and $k \equiv 1, -6, 5 \pmod{31}$ where $k = 1$ gives the cyclic group ($n_3 = 1$), $k \equiv 5$ and $k \equiv -6$ gives two non-abelian extensions.

□

1.11 General linear group

Exercise 1.11.1. Determine the range of the determinant when restricted to the following subgroups

a. $\mathrm{O}(n)$,

b. $\mathrm{U}(n)$,

c. $\mathrm{SP}_n(F)$.

Solution.

a. $\det \mathrm{O}(n) = \{\pm 1\}$ for $n \geq 2$, because the invariant inner product permits to define an orthonormal basis that defined a volume form after choosing an orientation (e.g. ordering the base). Now an arbitrary orthogonal transformation will map the orthogonal basis to another orthogonal basis but possibly change the orientation (for $n \geq 2$). Thus we obtain $(\wedge^n \delta)(\mathrm{vol}, g^*\mathrm{vol}) = \pm 1$ for every $g \in \mathrm{O}(n)$, δ the standard Euclidian inner product on $\mathbb{R}^n$ and vol the volumen form defined by the orthonormal basis (with the given orientation). Therefore $g^*\mathrm{vol} = \pm\mathrm{vol}$ the claimed result.

b. $\det \mathrm{U}(n) = \mathrm{U}(1) = \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, because we can choose again an ordered unitary base of $\mathbb{C}^n$ with the standard hermitean inner product h. The transform of the base under unitary transformations can

c. $\det \mathrm{SP}_n(F) = \{1\}$, because the preserved symplectic form $\omega = J\colon \wedge^2 F \to F$ induces a (preserved) volume form $\mathrm{vol} = \omega^{\wedge n}$ (Liouville's observation).

□

Exercise 1.11.2. Show that $\mathrm{O}(n)$ over $\mathbb{R}$ consists of at least 2 connected components. Conclude that the same is also true for $\mathrm{GL}_n(\mathbb{R})$.

Solution. According to the last exercise $\det\colon \mathrm{O}(n) \to \{\pm 1\}$. It is surjective, because $\det \begin{pmatrix} \pm 1 & \dots & & \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & \dots & & 1 \end{pmatrix} = \pm 1$. On the other hand $\{\pm 1\} \subset \mathbb{R}$ is not connected (and det continuous), so $\mathrm{O}(n)$ cannot be connected either. The elements of the connected component of $\mathbb{1}$ are called rotations while those of the other component are called reflections. Note that $-\mathbb{1}$ is a rotation for n even and a reflection for n odd.

The image of $\det\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*$ is the whole $\mathbb{R}^*$ ($\mathbb{R}^* \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ via the same upper left corner) which is also non-connected. □

1.12 Representations of finite groups

Exercise 1.12.1. Given a finite subgroup of $G \subset \mathrm{GL}(V)$ a real (or complex) vector space, show that

a. every $g \in G$ has determinant $\det g \in \Omega_*$ the group of roots of unity 1,

b. give an example of an element in $g \in \mathrm{GL}_2(\mathbb{R})$ that has finite order, but not determinant 1,

c. G is isomorphic to a subgroup of $\mathrm{O}(V)$ (or $\mathrm{U}(V)$ respectively).

Solution.

a. Given $\det g \notin \Omega^*$ that means that the image of $\langle g \rangle$ under the homomorphism $\det\colon \mathrm{GL}(V) \to F^*$ is infinite. But then $\langle g \rangle$ itself cannot be finite.

b. Consider the element $g := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ which has determinant -1 and $g^2 = \mathbb{1}$.

c. Given $\rho\colon G \hookrightarrow \mathrm{GL}(V)$ finite and V endowed with an positive definite inner product. Then the averaging as defined in class makes $\rho\colon G \hookrightarrow \mathrm{O}(V, \langle\!\langle ., . \rangle\!\rangle) \subset \mathrm{GL}(V)$ which is still an embedding. But $(V, \langle\!\langle ., . \rangle\!\rangle) \cong (V, \langle ., . \rangle)$, because they are both Euclidean vector/ Hilbert spaces of the same dimension. The same isomorphism makes G isomorphic to a subgroup $G' \subset \mathrm{O}(V, \langle ., . \rangle)$ (or $\mathrm{U}(V, \langle ., . \rangle$ respectively).

□

Exercise 1.12.2. Decompose the (left)-regular representation of $S_3 \oplus C_4$ over the complex numbers $\mathbb{C}$ into irreducible representations.

Solution. Note that $G := S_3 \oplus C_4$ and thus the (left)-regular representation is $\lambda_G = \tilde{\lambda}_{S_3} \otimes \tilde{\lambda}_{C_4}$ where λ_G stands for the (left)-regular representation of the (finite) discrete group G and $\tilde{\lambda}$ to its promotion to $S_3 \oplus C_4$ by forgetting about the other factor. As shown in class $\lambda_{S_3} = 1 \oplus \mathrm{sgn} \oplus \pi_2 \oplus \bar{\pi}_2$. Representation theory for C_4 is much simpler, because C_4 is cyclic. The representations are

ρ	$\chi\colon 0$	1	2	3	$\lvert\chi\rvert^2$	comment
λ_{C_4}	4	0	0	0	4	regular repr.
1	1	1	1	1	1	$\langle \lambda, 1 \rangle = 1$
sgn_4	1	-1	1	-1	1	$\langle \lambda, \mathrm{sgn}_4 \rangle = 1$
i	1	i	-1	$-\mathsf{i}$	1	$\langle \lambda, \mathsf{i} \rangle = 1$
$\bar{\mathsf{i}}$	1	$-\mathsf{i}$	-1	i	1	$\langle \lambda, \bar{\mathsf{i}} \rangle = 1$

Thus $\lambda_{C_4} = 1 \oplus \mathrm{sgn}_4 \oplus \mathsf{i} \oplus \bar{\mathsf{i}}$.

Note that the inner product on $L_2(C_4)$ is sesquilinear, i.e. $\langle \chi_1, \chi_2 \rangle = \frac{1}{4}(\bar{\chi}_1(0)\chi_2(0) + \cdots + \bar{\chi}_1(3)\chi_2(3))$ otherwise we could not have $\lvert\mathsf{i}\rvert^2 = 1$ which is clearly irreducible. Therefore $\langle \mathsf{i}, \bar{\mathsf{i}} \rangle = 0$, i.e. the two representations are not equivalent.

In total this gives

$$\begin{aligned}
\lambda_{S_3 \oplus C_4} &\cong \lambda_{S_3} \otimes \lambda_{C_4} \\
&= (1 \oplus \mathrm{sgn}_{S_3} \oplus \pi_2 \oplus \bar{\pi}_2) \otimes (1 \oplus \mathrm{sgn}_4 \oplus \mathsf{i} \oplus \bar{\mathsf{i}}) \\
&= 1 \oplus \mathrm{sgn}_4 \oplus \mathsf{i} \oplus \bar{\mathsf{i}} \oplus \mathrm{sgn}_{S_3} \oplus \pi_2 \oplus \bar{\pi}_2 \\
&\quad \oplus \mathrm{sgn}_{S_3} \mathrm{sgn}_{C_4} \oplus \ldots .
\end{aligned}$$

□

Exercise 1.12.3 (Representation induced by a finite group action). **a.** Simplify the definition of a group action (Defintion 1.8.1) on a finite space X in terms of a group homomorphism and a symmetric group.

b. Show that every action μ of G on a finite space X induces a representation on the finite-dimensional vector space $F[X] := \langle e_x : x \in X \rangle_F$.

c. What do orbits and fixed-points relate to?

Solution.

a. The definition of group action reduces to $\mu\colon G \to S(X)$ is a group homomorphism.

b. The induced representation is just $\rho\colon G \to \mathrm{GL}(F[X]) : g \mapsto (e_x \mapsto e_{\mu(g)x})$ and F-linearly extended.

c. Orbits correspond to invariant subspaces and fixed-points to joint eigenvectors/ $\sim$-spaces of eigenvalue 1.

$\square$