Abstract algebra: Homework 5 – Solutions

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1.8 Symmetric groups

Exercise 1.8.1 (2P).

a. Show that S_n is generated by (12), (23),..., $(n-1\, n)$.

b. Show that S_n is generated by (12) and $(12\dots n)$.

Solution. **a.** We know that S_n is generated by all permutations. If we could show that the permutation (ij) is a product of the above permutations, then we are done. Obviously $(12)(23)(12) = (13)$, i.e. we also know how to produce (13). By induction we can produce $(1k)$. Now $(1i)(1j)(1i) = (ij)$ and thus every S_n is generated by the (12), (23), ..., $(n-1, n)$.

b. According to the last part, it is sufficient to show that $(1i)$ can be generated by the two elements. For this notice that $(12)(123\dots n)(12) = (\dots)$

□

Exercise 1.8.2 (1P).

a. Show that $S_4 \cong \langle a, b : a^4 = \mathrm{id} = b^2 = (ba)^3\rangle$.

b. Show that $A_4 \cong \langle a, b : a^3 = \mathrm{id} = b^2, aba = ba^2b\rangle$.

Solution. **a.** We just need to find an element $a \in S_4$ of order 4 and another $b \in S_4$ of order 2, show that they fulfill the required relation $\mathrm{ord}(ab) = 3$ together with $\langle a, b\rangle = S_4$. So a must be any 4-cycle, e.g. $a = (1234)$ and according to the previous exercise $b = (12)$ guarantees that a and b together generate S_4. Note that $\mathrm{ord}(12) = 2$ and $ab = (134)$ has order 3. Therefore $\phi\colon \langle a, b : \dots\rangle \to S_4 : a \mapsto (1234), b \mapsto (12)$ can be extended to an isomorphism of the two groups.

b. We thus need elements a' of order 3 and b' of order 2 in S_4. An obvious choice would be $a = (123)$ and $b = (12)(34)$ with $ab = (134)$ ...

□

Exercise 1.8.3 (2P). How many elements of order k are there in S_n?

Solution. **a k-cycles.** Their number is $C_n(k) := \frac{n!}{k(n-k)!}$ for $2 \leq k \leq n$ or $C_n(1) = 1$, because the cycle $(a_2 a_3 \dots a_p a_1)$ is the same as $(a_1 a_2 \dots a_p)$.

$\square$

Exercise 1.8.4 (3P). Consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 5 & 6 & 4 & 2 & 8 & 3 & 1 \end{pmatrix}$

a. Write σ as product of disjoint orbits. Determine its signum and its order.

b. What is the order of the centralizer of σ in S_8? What is the order of the conjugacy class of σ?

Solution. **a.** $\sigma = (17368)(25), \operatorname{sgn}\sigma = -1, \operatorname{ord}\sigma = \operatorname{lcm}(\operatorname{ord}(17368), \operatorname{ord}(25)) = 10.$

b. $\operatorname{cent}_{S_8}\sigma = \operatorname{cent}_{S_8}(25) \cap \operatorname{cent}_{S_8}(17368) = \langle (25), (17368) \rangle$ and thus $(\operatorname{cent}_{S_8}\sigma : 1) = 10$, $|C(\sigma)| = (S_8 : \operatorname{cent}_{S_8}\sigma) = 8!/10$.

$\square$

Exercise 1.8.5 (2P).

a. List all conjugacy classes of S_5 together with their orders.

Solution. **a.** Two elements in S_n are conjugate iff they are of the same type. The type of a permutation is the length of its disjoint orbits (with multiplicity). Thus the orbit types are 1, 5, 4, 3, 2, $(2,2)$, and $(2,3)$. The order of an element in a conjugacy class of type $(n_1, \dots, n_k)$ is $\operatorname{lcm}(n_1, \dots, n_k)$. The number of elements are $|C_1| = 1$, $|C_5| = \frac{5!}{5} = 24$, $|C_4| = \frac{5!}{4} = 30$, $|C_3| = \frac{5!}{3 \cdot 2!} = 20$, $|C_2| = \binom{5}{2} = 10$, $|C_{(2,2)}| = \frac{5!}{2!(2!)^2} = 15$ and $|C_{(2,3)}| = \frac{5!}{2!3} = 20$ which indeed adds up to $5! = 120$.

$\square$

1.9 Sylow theorem

Exercise 1.9.1 (1P). Given a finite group whose order is divisible by a prime p. Show that there is a subgroup of order p (without using the detailed results of the first lemma in the proof of Sylow's theorem(s)).

Solution. Due to Sylow's theorem(s), there is a p-Sylow subgroup $S_p \subset G$ of order $p^K | (G : 1)$ where $\gcd((G : 1)/p^K, p) = 1$. But every element of S_p has order p^k with $0 \leq k \leq K$. if we have any element $a \in S_p$ of order p^k with $k \geq 1$, then $a^{p^{k-1}} \in S_p \subset G$ has order p and is thus the required element. $\square$

Exercise 1.9.2 (2P). Find the Sylow subgroups of

a. S_4 and

b. S_5.

Solution. **a.** $4! = 24 = 2^3 3$. Thus $S_p = \{\text{id}\}$ for $p \neq 2, 3$. In addition $n_3 \equiv 1 \pmod 3$ and $n_3|8$. But we know that the only nontrivial normal subgroup in S_4 is A_4, so $n_3 = 4$ and the 3-Sylow subgroups are $\langle(123)\rangle$, $\langle(124)\rangle$, $\langle(134)\rangle$, and $\langle(234)\rangle$. Finally $n_2 \equiv 1 \pmod 2$ and $n_2|3$ and thus $n_2 = 3$. The 2-Sylow subgroups are thus $\langle(1234), (13)\rangle$, $\langle(1324), (12)\rangle$ and $\langle\ldots\rangle$.

b. $5! = 120 = 2^3 \cdot 3 \cdot 5$ Thus $S_p = \{\text{id}\}$ for $p \neq 2, 3, 5$. Also $n_3 \equiv 1 \pmod 3$ and $n_3|40$ and the only nontrivial normal subgroup is A_5 of order 60, so $n_3 = 4, 10$ or 40. There are exactly $\binom{5}{3} = 10$, 3-Sylow subgroups generated by a 3-cycle each. And there are not enough letters for two disjoint 3-cycles, so $n_3 = 10$. For $p = 5$ we obtain $n_5 \equiv 1 \pmod 5$ and $n_5|24$ so $n_5 = 6 = \frac{5!}{5 \cdot 4}$, i.e. $\langle(12345)\rangle$, $\langle(12354)\rangle$, $\langle(12435)\rangle$, $\langle(12453)\rangle$, $\langle(12534)\rangle$, and $\langle(12543)\rangle$. Finally for $p = 2$ we have $n_2 \equiv 1 \pmod 2$ and $n_2|15$, so $n_2 = 3, 5$ or 15. We can reproduce the 3 examples of subgroups of order 8 by only using 4 out of 5 letters and thus end up with 15 different subgroups of order 8.

□

Exercise 1.9.3 (3P). Find all groups of order

a. 33,

b. 35,

c. 45.

Solution. **a.** $33 = 3 \cdot 11$ so $n_3 \equiv 1 \pmod 3$ and $n_3|11$ which only leaves $n_3 = 1$, i.e. there is a unique normal subgroup $S_3 \triangleleft G$ of order 3. Conversely $n_{11} \equiv 1$ (mod 1)1 and $n_{11}|3$, i.e. there is another unique normal subgroup $S_{11} \triangleleft G$ of order 11. Since their only common element is of order 1, we have $G = S_3 \times S_1 \cong \mathbb{Z}/(33)$. I.e. all the groups of order 33 are cyclic.

b. $35 = 5 \cdot 7$ and correspondingly $n_5 \equiv 1 \pmod 5$ and $n_5|7$ which leaves $n_5 = 1$ and thus $S_5 \triangleleft G$ the unique subgroup of order 5 which is normal. Conversely $n_7 \equiv 1 \pmod 7$ and $n_7|5$ which leads to $S_7 \triangleleft G$ is the unique subgroup of order 7 and thus normal. Therefore correspondingly $G = S_5 S_7 \cong \mathbb{Z}/(35)$ is cyclic.

c. $45 = 3^2 \cdot 5$ and so $n_3 \equiv 1 \pmod 3$ with $n_3|5$ leads to $n_3 = 1$ and thus $S_3 \triangleleft G$ is the unique subgroup of order 9 which is therefore normal. $n_5 \equiv 1 \pmod 5$ and $n_5|9$ also leads to $n_5 = 1$ and thus $S_5 \triangleleft G$ is the only subgroup of order 5. Therefore there are 2 non-equivalent groups of order 45, $\mathbb{Z}/(45)$ and $(\mathbb{Z}/(3))^2 \times \mathbb{Z}/(5)$ both of which are abelian.

□

Exercise 1.9.4 (4P). Show that the following are not simple groups, i.e. that they have a non-trivial normal subgroup:

a. A group of order 18;

b. A group of order 30;

c. A group of order 56.

Solution. **a.** $18 = 2 \cdot 3^2$ and thus $n_3 \equiv 1 \pmod 3$ with $n_3|2$ which implies $n_3 = 1$ and thus $S_3 \triangleleft G$ is the unique subgroup of order 9 which is normal. So G is not simple.

b. $30 = 2 \cdot 3 \cdot 5$ and thus $n_5 \equiv 1 \pmod 5$ with $n_5|6$. If there is only one subgroup $S_5 \subset G$ of order 5, then it is normal and thus G not simple. Consider thus the case $n_5 = 6$. Conversely $n_3 \equiv 1 \pmod 3$ and $n_3|10$. Again if there is only one subgroup $S_3 \subset G$ of order 3, then it is normal and thus G not simple. Consider thus the case $n_3 = 10$. But that already leads into a contradiction because 6 different subgroups of order 5 have 24 elements beside the unit and 10 different subgroups of oder 3 have 20 elements beside the unit which must all be different from the elements in the S_5, because they have order 3 and not 5. But G has only 29 elements beside the unit. So this case does not occur. In total we arrive at G not being simple.

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