Homework 2, Solutions

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1.3 Isomorphism theorems

Exercise 1.3.1. Let $\phi\colon A \to B$ and $\psi\colon A \to C$ be group homomorphisms. Prove the following: If ψ is surjective, then ϕ factors through ψ if and only if $\ker\psi \subset \ker\phi$. In this case ϕ factors uniquely through ψ.

Solution. Since ψ is surjective we can hope that it is a projection. Namely let $N := \ker\psi$ be its kernel. Then clearly ψ factors through $\pi_N\colon A \to A/N$ via $\bar{\psi}\colon A/N \xrightarrow{\sim} C$ as $\psi = \bar{\psi} \circ \pi_N$. But since $\ker\psi = N = \ker\pi_N$, $\bar{\psi}$ is injective. Also since ψ is surjective, so is $\bar{\psi}$. Therefore $\bar{\psi}$ is an isomorphism and in particular $\pi_N = (\bar{\psi}^{-1} \circ \psi)\colon A \to A/N$, i.e. ψ is as good as the projection (just needs to be mapped with an isomorphism).

But then the Proposition 1.3.1 shows that ϕ factors uniquely through ψ if $\ker\phi \supset N = \ker\psi$. Conversely for $N \supsetneq \ker\phi$ there is an $a \in N \setminus \ker\phi$ that will be mapped to id_C via ψ even though $\phi(a) \neq \mathrm{id}_B$. But then $\phi(a) \neq \mathrm{id}_B = (\bar{\phi} \circ \psi)(a)$ for any $\bar{\phi}$. This completes the proof. □

Exercise 1.3.2. Show that the identity homomorphism $\mathrm{Id}\colon 2\mathbb{Z} \xrightarrow{\sim} 2\mathbb{Z}$ does not factor through the inclusion homomorphism $\imath\colon 2\mathbb{Z} \hookrightarrow \mathbb{Z}$ even though $\ker\imath \subset \ker\mathrm{Id}$.

Hint: Opposite to the situation in Exercise 1.3.1, $\imath$ is not surjective.

Solution. Assume Id would factor through $\imath$. Then there would be a homomorphism $\bar{\mathrm{Id}}\colon \mathbb{Z} \to 2\mathbb{Z}$ such that $\bar{\mathrm{Id}} \circ \imath = \mathrm{Id}$. Since Id is surjective, so must $\bar{\mathrm{Id}}$ be. Thus $\bar{\mathrm{Id}}(1) = \pm 2$. On the other hand $\imath(2) = 2$ which implies $(\bar{\mathrm{Id}} \circ \imath)(2) = \pm 4 \neq 2 = \mathrm{Id}(2)$. Therefore Id cannot factor through $\imath$. □

Exercise 1.3.3. Let $\phi\colon A \to C$ and $\psi\colon B \to C$ be group homomorphisms. Prove the following: If ψ is injective, then ϕ factors through ψ if and only if $\mathrm{im}\,\phi \subset \mathrm{im}\,\psi$. In this case ϕ factors uniquely through ψ.

Solution. We want to find a $\bar{\phi}\colon A \to B$ such that $\phi = \psi \circ \bar{\phi}$. The obvious choice is $\bar{\phi}\colon a \mapsto b \in B$ whith $\psi(b) = \phi(a)$. Since ψ is injective, there is at most one such b. But since also $\operatorname{im}\phi \subset \operatorname{im}\psi$ we know that there is at least one $b \in B$ with that property. It remains to show that $\bar{\phi}$ is a homomorphism. Let thus $a_i \in A$. We know that there are (unique) $b_i \in B$ with $\psi(b_i) = \phi(a_i)$. But then also $\psi(b_1b_2) = \psi(b_1)\psi(b_2) = \phi(a_1)\phi(a_2) = \phi(a_1a_2)$, i.e. $\bar{\phi}(a_1a_2) = b_1b_2$ is the unique choice. But this is $b_1b_2 = \bar{\phi}(a_1)\bar{\phi}(a_2)$ and therefore $\bar{\phi}$ a homomorphism. □

Remark (1.4.99)**.** The corresponding counter-example in the non-injective case is, e.g. $\mathrm{Id}\colon \mathbb{Z}/(2) \xrightarrow{\sim} \mathbb{Z}/(2)$ which does not factor through $p\colon \mathbb{Z} \to \mathbb{Z}/(2)$ even though $\operatorname{im}\mathrm{Id} = \mathbb{Z}/(2) \subseteq \mathbb{Z}/(2) = \operatorname{im} p$. This is also easy to see, because the only homomorphism $\mathbb{Z}/(2) \to \mathbb{Z}$ is the trivial map $\phi_0\colon a \mapsto 0$, but obviously $p \circ \phi_0 \neq \mathrm{Id}$.

Exercise 1.3.4. Show that every subgroup of a cyclic group is cyclic.

Solution. Let $G \cong C_n$ and $g \in G$ a generator. For every subgroup $H \subset G$ we can consider all the elements $X = \{\log_g h : h \in H\} \subset \mathbb{Z}/(n)$. Let $d \in \mathbb{N}$ be their greatest common divisor. Due to the Euclidean algorithm we can write d as a finite linear combination of elements of X. But this means that we can write h_0 as a product of elements of $H \subset G$ which is abelian. Since d is a common divisor of X, we can write every element of X as a multiple of d, hence write every $h \in H$ as a power of h_0. Conversely all powers of h_0 must be contained in H, since H is a group. Therefore $H = \langle h_0 \rangle$, i.e. cyclic. □

Exercise 1.3.5. a. Show that the additive group $\mathbb{R}/\mathbb{Z}$ is isomorphic to the multiplicative group of all complex numbers $\mathbb{C}$ of modulus 1.

b. Show that the additive group $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the group of all complex roots of unity (i.e. all complex numbers $z \neq 0$ such that $\langle z \rangle$ is finite in $\mathbb{C}^*$).

c. Show that the complex n-th roots of unity $\Omega_n := \{z \in \mathbb{C} : z^n = 1\}$ form a cyclic group (w.r.t. multiplication).

Solution.

a. Consider first the homomorphisms $\phi_m\colon (\mathbb{R}, +) \to \mathbb{C}^*$. It is easy to see that these must be exponential maps, i.e. for every $m \in \mathbb{C}$, $\phi_m(t) := \exp(mt)$ gives a group homomorphism. If we want $\operatorname{im}\phi_m \subset \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$, then we need $m = \mu\mathrm{i}$ for some $\mu \in \mathbb{R}$. If we want moreover equality, we need $\mu \neq 0$. In addition we want ϕ_m to

factor through $\mathbb{Z}$, so $\exp(\mu\mathsf{i}) = 1$, i.e. $\mu = 2\pi n$ for $n \in \mathbb{Z} \setminus 0$. Then $\bar{\phi}_n \colon \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1 : t + \mathbb{Z} \mapsto \exp(2\pi\mathsf{i}nt)$. But then the kernel in $\mathbb{R}/\mathbb{Z}$ is $\langle \frac{1}{n} + \mathbb{Z} \rangle$, i.e. the only two isomorphisms are $\bar{\phi}_{\pm 1}(t) = \exp(\pm 2\pi\mathsf{i}t)$. Therefore $(\mathbb{R}/\mathbb{Z}, +) \cong \mathbb{S}^1 \subset \mathbb{C}^*$.

b. Each of the two isomorphisms also maps $\mathbb{Q}/\mathbb{Z}$ isomorphic to the subset of the torsion elements $\mathrm{Tor}(\mathbb{C}^*) = \{z \in \mathbb{S}^1 : \exists n > 1 : z^n = 1\}$,[1] because all elements in $\mathbb{Q}/\mathbb{Z}$ can be written in the form $\frac{p}{q} + \mathbb{Z}$ with $\gcd(p, q) = 1$ and $q > 0$ and are therefore of finite order q. Conversely every element of order q in $\mathbb{R}/\mathbb{Z}$ must be of the form $\frac{\alpha}{q} + \mathbb{Z}$ with $\alpha \in \mathbb{Z}$ and $\gcd(\alpha, q) = 1$.

c. In particular the n-th roots of unity are the images of $\frac{k}{n} + \mathbb{Z}$ with $k \in \mathbb{Z}/(n)$ the only elements of $\mathbb{R}/\mathbb{Z}$ with exponent n (i.e. $n \cdot g = 0 \in \mathbb{R}/\mathbb{Z}$). The images are $\exp(2\pi\mathsf{i}k/n)$ where k attains one representative of each coset of $\mathbb{Z}/(n)$. Therefore the n-th roots of unity are generated (as a multiplicative group) by a primitive n-th root, e.g. $\omega_n = \exp(\pm 2\pi\mathsf{i}/n)$.

□

Exercise 1.3.6. Consider the group $D_4 := \langle \sigma, \tau : \sigma^2 = \mathrm{id} = \tau^4, \sigma\tau\sigma = \tau^{-1} \rangle$

a. Find the order of every element in D_4,

b. Show that for every $d | (D_4 : 1)$ there is a subgroup $S \subset D_4$ of order d.

Solution.

a. The elements of D_4 have order 1: id; 2: $\tau^2, \sigma\tau^k$; and 4: τ, τ^{-1}, which totals to 8 the order of D_4.

b. 1: the only solution is $\{\mathrm{id}\} = \langle \emptyset \rangle$; 2: $\langle \sigma\tau^k \rangle$ or $\langle \tau^2 \rangle$; 4: $\langle \tau \rangle$; 8: D_4.

□

Exercise 1.3.7. a. Let G be a finite group and $S, T \subset G$ any subgroups. Show that $|ST| = |S|\,|T|/|S \cap T|$.

b. Find a group G together with subgroups $S, T \subset G$ such that $ST \subset G$ is not a group.

Solution (as found by the students fall 2012).

[1] $z \in \mathrm{Tor}(\mathbb{C}^*)$ implies $|z| = 1$, because $1 = z^n$ implies $1 = |z|^n$.

a. Since $S \cap T \subset S$ is a subgroup, we have $|S|/|S \cap T| = n \in \mathbb{N}_+$. In particular there are n elements $s_i \in S$ such that $S = s_1(S \cap T) \cup \cdots \cup s_n(S \cap T)$. You can modify the s_i such that $s_i s_j^{-1} \notin T$ for $i \neq j$. But then also $ST = s_1 T \cup \ldots s_n T$ with $s_i T \cap s_j T = \emptyset$. This tells us that $|ST| = n|T| = |S|\,|T|/|S \cap T|$ as required.

b. Note that the above proof did not use that ST would be a subgroup. Indeed for $G = S_3$ and $S = \langle(12)\rangle$, $T = \langle(13)\rangle$ we see that $ST = \{\mathrm{id}, (12), (13), (132)\}$ which is not a group, because $(132)^{-1} = (123) \notin ST \neq TS$. □

Exercise 1.3.8. Let G be a finite group, $N \triangleleft G$ a normal subgroup and $H \subset G$ any subgroup such that $|N|$ and $(G : N)$ are relatively prime. Show that $H \subset N$ iff $|H|$ divides $|N|$.

Hint: Consider $g \in G$ with $\operatorname{ord} g \not| (N : 1)$.

Solution. First note that $\gcd(|N|, (G : N)) = 1$ can be rewritten as $(G : 1) = mn$, $(N : 1) = n$, $(G : N) = m$, and $\gcd(m, n) = 1$. Therefore $g \in G$ is in N iff $\operatorname{ord} g | n$, i.e. $g^n = \mathrm{id}$. Conversely every $h \in H$ has $\operatorname{ord} h | (H : 1)$. Therefore $(H : 1) | (N : 1)$ is sufficient. To see that it is also necessary, note that for every prime divisor $p | (H : 1)$ there is an $h \in H$ with $\operatorname{ord} h = p$, i.e. if $(H : 1)$ contains a prime factor of m, then there is an element $h \in H$ that is not in N. □

1.4 Free groups, free products, and presentations

Exercise 1.4.1. Given a group G, the conjugates of an element $x \in G$ are $C_x := \{gxg^{-1} : g \in G\}$. Given a subset $S \subset G$, there exists a smallest normal subgroup $N \triangleleft G$ that contains $S \subset N$. Show that N consists of all products of elements in $C_{S \cup S^{-1}}$.

Solution. Remember that a normal subgroup $N \triangleleft G$ fulfills $gNg^{-1} \subset N$ for all $g \in G$. The smallest normal subgroup containing S is therefore $N_1 := \langle S \rangle_G = \bigcap_{S \subset N \triangleleft G} N$ which is indeed contained in all $N \triangleleft G$ with $S \subset N$. Let conversely $N_2 = \langle gsg^{-1}, gs^{-1}g^{-1} : s \in S, g \in G \rangle$ be the subgroup generated by the conjugacy classes of elements of S and of $\bar{S} := S^{-1} := \{s^{-1} : s \in S\}$. Obviously $N_2 \subset N_1$, because $S, \bar{S} \subset N_1$ and N_1 is closed under conjugation and a group.

If we can show that N_2 is a group, then also $N_1 \subset N_2$, because $c_g(abc) = gabcg^{-1} = c_g(a)c_g(b)c_g(c)$, i.e. conjugation is a group-automorphism, and $c_g \circ c_h = c_{gh}$, i.e. $c_\bullet \colon G \to \mathrm{Aut}(G)$ a homomorphism. The latter two together mean that N_2 is stable under conjugation. Obviously $\mathrm{id} \in N_2$, the product of 0 generators. The general element of N_2 is $n = c_{g_1}(s_1) \dots c_{g_n}(s_n)$ with $g_i \in G$ and $s_i \in S \cup \bar{S}$. Then also the inverse $n^{-1} = g_n^{-1} s_n^{-1} g_n \dots g_1 = c_{g_n^{-1}}(s_n^{-1}) \dots c_{g_1^{-1}}(s_1)$ is an element of N_2. Therefore N_2 is one of the normal subgroups $N \triangleleft G$ that contain $S \subset N$ all of whose intersection is N_1, and so $N_1 \subset N_2$. So $N_1 = N_2$ which completes the proof. □

Exercise 1.4.2. a. List (compactly) all elements of the group $\langle a, b : a^2 = \mathrm{id} = b^2 \rangle$. Give a compact multiplication table of the group.

b. List all elements of the group $\langle a, b : a^2 = \mathrm{id} = b^2 = (ab)^3 \rangle$ and give their multiplication table. Which known group is it isomorphic to?

Solution.

a. Note that we can cancel every two consecutive a or two consecutive b. Therefore we are left with terms like $(ab)^n$ or $(ba)^n$ with an optional a or b in the end, i.e. $\{(ab)^n, (ab)^n a, (ba)^n, (ba)^n b : n \in \mathbb{N}\}$. Note that $(ab)^0 = \mathrm{id} = (ba)^0$, but that is the only ambiguity in the table. The only cancellation rule is the one mentioned before. Therefore the multiplication table is

id	$(ab)^n$	$(ab)^n a$	$(ba)^n$	$($
$(ab)^m$	$(ab)^{m+n}$	$(ab)^{m+n}a$	$\begin{cases} (ab)^{m-n} & m \geq n, \\ (ba)^{n-m} \end{cases}$	$\begin{cases} (ab)^{m-n} \\ (ba)^{n-} \end{cases}$
$(ab)^m a$	$\begin{cases} (ab)^{m-n}a & m \geq n, \\ (ba)^{n-m-1}b \end{cases}$	$\begin{cases} (ab)^{m-n}a & m \geq n, \\ (ba)^{n-m-1} \end{cases}$	$(ab)^{m+n}a$	$(ab$
$(ba)^m$	$\begin{cases} (ba)^{m-n} & m \geq n, \\ (ab)^{n-m} \end{cases}$	$\begin{cases} (ba)^{m-n-1}b & m > n, \\ (ab)^{n-m}a \end{cases}$	$(ba)^{m+n}$	$(ba$
$(ba)^m b$	$(ba)^{m+n}b$	$(ba)^{m+n+1}$	$\begin{cases} (ba)^{m-n}b & m \geq n, \\ (ab)^{n-m-1}a \end{cases}$	$\begin{cases} (ba)^{m-} \\ (ab)^{n-} \end{cases}$

b. Maybe it is easier to start with this second part, because there are only finitely many elements. Basically we start with all the elements of part a and cancel the duplicates. Obviously $0 \leq n < 3$, but also $aba = bab$. This leads to the elements $\{\mathrm{id}, a, b, ab, ba, aba\}$. The multiplication

table is now

id	a	b	ab	ba	$aba = bab$
a	id	ab	b	aba	ba
b	ba	id	aba	a	ab
ab	aba	a	ba	id	b
ba	b	aba	id	ab	a
aba	ab	ba	a	b	id

which is the (only) non-Abelian group $D_3 = S_3$ of order six.[2] □

Remark (1.4.99). Note that $\langle a : a^2 = \mathrm{id}\rangle \cong C_2 \cong \langle b : b^2 = \mathrm{id}\rangle$ are cyclic groups of order 2. Since the group in part a only fulfills the two relations $a^2 = \mathrm{id} = b^2$, it is (isomorphic to) the free product $C_2 * C_2$.

Exercise 1.4.3. The multiplication of the unit quaternions $\mathsf{i}^2 = -1 = \mathsf{j}^2 = \mathsf{k}^2$, $\mathsf{ij} = \mathsf{k} = -\mathsf{ji}$, $\mathsf{jk} = \mathsf{i} = -\mathsf{kj}$, $\mathsf{ki} = \mathsf{j} = -\mathsf{ik}$ together with $\mathbb{R}$-linearity implies for $a, b, c, d, a', b', c', d' \in \mathbb{R}$,

$$\begin{aligned}(a + b\mathsf{i} + c\mathsf{j} + d\mathsf{k})(a' + b'\mathsf{i} + c'\mathsf{j} + d'\mathsf{k}) = \\ = (aa' - bb' - cc' - dd') + (ab' + ba' + cd' - dc')\mathsf{i} \\ + (ac' + ca' + db' - bd')\mathsf{j} + (ad' + da' + bc' - cb')\mathsf{k}.\end{aligned}$$

a. Show that the multiplication is associative.

b. Let $\overline{a + b\mathsf{i} + c\mathsf{j} + d\mathsf{k}} := a - b\mathsf{i} - c\mathsf{j} - d\mathsf{k}$ and $|z|^2 := z\bar{z}$ for every quaternion $z \in \mathbb{H}$. Show that $|z_1 z_2| = |z_1|\,|z_2|$ for every pair of quaternions $z_{1/2} \in \mathbb{H}$.

c. Conclude that $\mathbb{H}^* := \mathbb{H} \setminus \{0\}$ is a group under multiplication. (What is the inverse? Therefore $\mathbb{H}$ is called a division algebra.)

Solution.

a. The brute-force method is to multiply out, i.e. let $a_n, b_n, c_n, d_n \in \mathbb{R}$ be real numbers for $n = 1, 2, 3$ and consider the three quaternions $z_n := a_n + b_n\mathsf{i} + c_n\mathsf{j} + d_n\mathsf{k}$ and the two possible ways to multiply $z_1 z_2 z_3$

[2] e.g. via $a \mapsto \sigma, b \mapsto \sigma\tau$ and thus indeed $ab \mapsto \tau$ of order 3.

in that order.

$$
\begin{aligned}
&(z_1z_2)z_3\\
&= \big((a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\mathsf{i}\\
&\quad + (a_1c_2 + c_1a_2 + d_1b_2 - b_1d_2)\mathsf{j} + (a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)\mathsf{k}\big)(a_3 + b_3\mathsf{i} + c_3\mathsf{j} + d_3\mathsf{k})\\
&= a_1a_2a_3 + \cdots + (\cdots + b_1d_2b_3)\mathsf{k}\\
&= (a_1 + b_1\mathsf{i} + c_1\mathsf{j} + d_1\mathsf{k})\big((a_2a_2 - b_2b_3 - c_2c_3 - d_2d_3)\\
&\quad + (a_2b_2 + b_2a_3 + c_2d_3 - d_2c_3)\mathsf{i} + (a_2c_3 + c_2a_3 + d_2b_3 - b_2d_3)\mathsf{j} + (a_2d_3 + d_2a_3 + b_2c_3 - c_2b_3)\\
&= z_1(z_2z_3)
\end{aligned}
$$

The problem is that the 4^3 terms in the middle are really hard to control. It would be much easier if we broke up the test into several simpler tests. Due to $\mathbb{R}$-linearity it would be sufficient to check associativity for the 3 elements $\mathsf{i}, \mathsf{j}, \mathsf{k}$ of Q. But since Q is generated by i and j alone, it is also sufficient to check for these two. Finally, $\mathbb{R}$-linearity reduces further to multiplication with the elements i, j, and k. Thus we have to check 2 tables with 3×3 entries each. As verification we will give the 9 elements in each table (according to the Libby test from Section 1.1). The multiplication table reads:

1	i	j	k
i	-1	k	$-\mathsf{j}$
j	$-\mathsf{k}$	-1	i
k	j	$-\mathsf{i}$	-1

Therefore the two tables are:

i	-1	k	$-\mathsf{j}$
-1	$-\mathsf{i}$	$-\mathsf{j}$	$-\mathsf{k}$
$-\mathsf{k}$	$-\mathsf{j}$	i	1
j	$-\mathsf{k}$	-1	i

j	$-\mathsf{k}$	-1	i
k	j	$-\mathsf{i}$	-1
-1	$-\mathsf{i}$	$-\mathsf{j}$	$-\mathsf{k}$
$-\mathsf{i}$	1	$-\mathsf{k}$	j

Thus Q passes the associativity test. And therefore also the (reduced) group algebra is associative.

b. We know that $|z| \geq 0$ is unique once $|z|^2 \geq 0$ is known. We try therefore to show that $|z_1z_2|^2 = |z_1|^2|z_2|^2$. Again the brute-force method gives 64 terms which may be hard to oversee.

Instead we use

$$|z_1z_2|^2 = \langle z_1z_2, z_1z_2\rangle = \overline{z_1z_2}z_1z_2 = \bar{z}_2(\bar{z}_1z_1)z_2 = |z_1|^2\bar{z}_2z_2 = |z_1|^2|z_2|^2$$

where we have used in addition

$$\begin{aligned}\overline{z_1 z_2} &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) - (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\mathsf{i} \\ &\quad - (a_1c_2 + c_1a_2 + d_1b_2 - b_1d_2)\mathsf{j} - (a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)\mathsf{k} \\ &= \bar{z}_2\bar{z}_1\end{aligned}$$

c. We therefore know that $z_1 z_2 = 0$ iff $z_1 = 0$ or $z_2 = 0$ (or both). Thus $\mathbb{H}$ is a non-commutative integral ring. The neutral element of multiplication is obviously $1 \in \mathbb{R} \subset \mathbb{H}$. Since $z\bar{z} = |z|^2 = \bar{z}z$ the inverse element of $z \neq 0$ must be $\frac{1}{|z|^2}\bar{z}$. Note that the division also depends on the order, i.e. instead of writing $\frac{z}{w}$ we should write ${}^{z}\!/_{w}$ or ${}_{w}\backslash^{z}$, depending on what quotient we need. □

Remark (1.5.99). It is possible to consider $\bar{z}z$ as a positive definite sesquilinear-form, namely $\langle z_1, z_2\rangle := \bar{z}_1 z_2$ which maps $\langle .,.\rangle \colon \mathbb{H} \times \mathbb{H} \to \mathbb{H}$, has

$$\langle z, w_1 + \lambda w_2\rangle = \langle z, w_1\rangle + \lambda\langle z, w_2\rangle$$

for all $w_i \in \mathbb{H}$ and $\lambda \in \mathbb{R}$,

$$\begin{aligned}\langle z_2, z_1\rangle &= (a_2a_1 + b_2b_1 + c_2c_1 + d_2d_1) + (a_2b_1 - b_2a_1 - c_2d_1 + d_2c_1)\mathsf{i} \\ &\quad + (a_2c_1 - c_2a_1 - d_2b_1 + b_2d_1)\mathsf{j} + (a_2d_1 - d_2a_1 - b_2c_1 + c_2b_1)\mathsf{k} \\ &= \overline{\langle z_1, z_2\rangle}, \\ |z|^2 := \langle z, z\rangle &= a^2 + b^2 + c^2 + d^2 \geq 0 \quad \text{and “}= 0\text{” iff } z = 0.\end{aligned}$$

I.e. $\langle .,.\rangle$ is indeed positive definite and sesquilinear.