Abstract Algebra – I Groups (群理论)

Melchior Grützmann / 古梅西

November 1, 2012

Outline

Group actions (群作用) I

Remember the introducing example of symmetries of the equilateral triangle. It turned out that the group permutes the corners of the triangle. This concept can be generalized to arbitrary groups in the following way.

Definition

Given a group G and a set (集合) X. An action (作用) *of G on X is a binary operation* $\mu\colon G \times X \to X : (g, x) \mapsto \mu(g)x$ *such that* $\mu(\mathrm{id}) = \mathrm{Id}_X$ *and* $\mu(gh) = \mu(g)\mu(h)$.
We denote $Gx := \{gx : g \in G\}$ *the* orbit (轨道) *of G through* $x \in X$ *and* $\mathrm{Stab}_G(x) := \{g \in G : \mu(g)x = x\}$ *the* stabilizer (稳定子群) *of* $x \in X$.
We say that a group G acts transitively (可递的作用) *on a set X if for every pair of elements* $x, y \in X$ *in X there is a group element* $g \in G$ *such that* $\mu(g)x = y$.

Group actions (群作用) II

Remark

By definition the group acts transitively on each orbit. Therefore intersecting orbits must coincide. Also transitivity from x to y is an equivalence relation (reflexivity $x \sim x$ via $\mathrm{id} \in G$, $y \sim x$ via g^{-1} if $x \sim y$ via $g \in G$, and transitivity $x \sim z$ via hg if $x \sim y$ via g and $y \sim z$ via $h \in G$.).

Remark

Correspondingly there is also the notion of a *right action (从右边的作用)* $\rho\colon X \times G \to X : (x, g) \mapsto x^g$. Note that this means in particular $\rho(gh) = \rho(h)\rho(g)$, i.e. $\rho\colon G \to S(X)$ is an anti-homomorphism. There is however a 1:1-correspondence between (left)- and right-actions via $\mu(g) := \rho(g^{-1})$.

Group actions (群作用) III

Corollary (of Lagrange's theorem)

Given the action of a finite group G on a set X, then for every $x \in X$ we have $\operatorname{ord} G = |Gx| \operatorname{ord} \operatorname{Stab}_G(x)$.

Proof.

Note that the orbit of G through $x \in X$ is isomorphic to the left-coset $G/\operatorname{Stab}_G(x)$. □

Example

Remember the conjugation action of a group G on itself: $c\colon G \to \mathrm{Aut}(G) : g \mapsto c_g$ with $c_g\colon G \to G : h \mapsto ghg^{-1}$. Clearly $c\colon G \to \mathrm{End}(G)$ is a monoid homomorphism, because $c_g(c_{g'}(h)) = gg'hg'^{-1}g^{-1} = (gg')h(gg')^{-1} = c_{gg'}(g)$, also $(c_{\mathrm{id}}\colon G \to G : h \mapsto \mathrm{id}\,h\,\mathrm{id}^{-1} = h) = \mathrm{Id}_G$ and thus $c_g^{-1} = c_{g^{-1}}$ is the inverse endomorphism and thus c_g indeed an automorphism for every $g \in G$.

We call the orbits of $x \in G$ under c the conjugacy classes (共轭类) of x. The conjugacy class of a central element $z \in \mathrm{cent}(G)$ is just $\{z\}$.

The partition into conjugacy classes of a finite group gives a partition of the group order, i.e.

$$|G| = |\,\mathrm{cent}\,G| + \sum_{|C|>1} |C|$$

where the sum runs over the non-trivial conjugacy classes of G. This is called the class equation of the group (群的类方程).

Example

0. for an abelian group $G = \operatorname{cent} G$ and thus the sum part is 0.
1. Remember the symmetry group of the square $D_4 = \langle \sigma, \tau : \sigma^2 = \mathrm{id} = \tau^4, \sigma\tau\sigma = \tau^{-1} \rangle$. Its center is $\operatorname{cent} D_4 = \{\mathrm{id}, \tau^2\}$ and the other two rotations form a conjugacy class$\{\tau, \tau^{-1}\}$.[1] The remaining reflections break into two conjugacy classes $\{\sigma\tau, \sigma\tau^{-1}\}$ and $\{\sigma, \sigma\tau^2\}$.[2] The class equation is therefore

$$|D_4| = 8 = 2 + 2 + 2 + 2.$$

[1] because $\sigma\tau\sigma = \tau^{-1}$

[2] because $\tau\sigma\tau^{-1} = \sigma\tau^{-2} = \sigma\tau^2$ and $\sigma(\sigma\tau)\sigma = \tau\sigma = \sigma\tau^{-1}$

Exercise

Explain how the original statement of Lagrange's theorem "*When $x_1, \ldots, x_n$ are permuted in all possible ways, then the number of different values of $f(x_1, \ldots, x_n)$ is a divisor of $n!$.*" relates to orbits and stabilizers.

Exercise

Let G be a group and for $g \in G$ define the inner automorphism (内自同构) $c_g \colon G \to G : h \mapsto ghg^{-1}$.

a. Show that the inner automorphisms c_g form a subgroup $\mathrm{Int}(G) \subset \mathrm{Aut}(G)$ isomorphic to $G/\,\mathrm{cent}(G)$.

b. Show that $\mathrm{Int}(G) \lhd \mathrm{Aut}(G)$, i.e. a normal subgroup.
   Hint: What is $(\phi \circ c_g)(h)$ for $g, h \in G$?

Exercise

Let $\mu\colon G \times X \to X$ be the action of a group G on a set X.

a. Let $x, y \in X$ be two points on the same orbit. Show that their stabilizers are conjugate, i.e. there is an element $g \in G$ such that $\mathrm{Stab}_G(x) = g\,\mathrm{Stab}_G(y) g^{-1}$.

b. Assume that $\mathrm{Stab}_G(x) \cong C_2$ and $\mathrm{Stab}_G(y) \cong C_3$. Can x and y be on the same orbit? (Justify your answer.)

Exercise

Show that in a finite group G of order n, an element of order k has at most n/k conjugates.

Exercise

Determine the class equation of the $D_n := \langle \sigma, \tau : \sigma^2 = \mathrm{id} = \tau^n, \sigma\tau\sigma^{-1} = \tau^{-1} \rangle$ where $n = 1, 2, \ldots$.

Exercise

Assume that $G/\operatorname{cent}(G)$ is cyclic. Prove that G is abelian.

Exercise

A characteristic subgroup (特征子群) $H \subset G$ is a subgroup that is invariant under all automorphisms, i.e. for all $\phi \in \operatorname{Aut}(G)$: $\phi(H) = H$. In particular characteristic subgroups are invariant under the inner automorphisms and therefore normal.

a. Show that the center $\operatorname{cent}(G)$ is a characteristic subgroup.
b. Prove that every characteristic subgroup $H \subset N$ of a normal subgroup $N \triangleleft G$ is normal $H \triangleleft G$ in G.
c. Assume that $N \triangleleft G$ is characteristic and $N \subset H \subset G$ with $H/N \subset G/N$ characteristic. Show that $H \subset G$ is characteristic.