# Abstract Algebra – I Groups (群理论)
# 1.12 Group representations (群表示论)

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# Outline

# Group representations (群表示论)

## Definition

*Given a discrete group $G$, a representation of $G$ on a vector space $V$ is a group homomorphism $\rho\colon G \to \mathrm{GL}(V)$.*

Examples 0. Given any goup $G$ and any vector space $V$, we can map $\rho\colon G \to \mathrm{GL}(V) : g \mapsto \mathbb{1}$, i.e. all elements are mapped to the identity of $V$. This is called *a* trivial representation. More particularly $V = 0$ and thus $\mathrm{GL}(V) = \{\mathbb{1}\}$ gives *the trivial representation*.

# Examples I

1. Given a finite group $G$ we define the vector space $F[G]$ with base $\{e_g : g \in G\}$ and the representation $\lambda\colon G \to \mathrm{GL}(F[G]) : g \mapsto (e_h \mapsto e_{gh})$, called the (left) regular representation (规则表示). As a more particular example consider $G = \mathbb{Z}/(3)$ and thus $V = F[\mathbb{Z}/(3)] \cong \mathbb{R}^3$,

$$\lambda(0) = \mathbb{1}, \quad \lambda(1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda(2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

# Examples II

2. Given two representations $\pi_i\colon G \to \mathrm{GL}(V_i)$ of a group $G$, then we can construct a new representation on $V := V_1 \oplus V_2$, $\pi\colon G \to \mathrm{GL}(V_1 \oplus V_2) : g \mapsto \pi_1(g) \oplus \pi_2(g)$ where the matrices are written diagonally above each other.
   A representation is now called irreducible (不可约表示) if it is nontrivial and cannot be written as the direct sum of two nontrivial representations. Conversely a representation is called fully reducible if it can be written as the direct sum of irreducible representations.

# Examples III

3. Given two representations $\pi_i\colon G \to \mathrm{GL}(V)$ of a group $G$, then we can construct a new representation on $V := V_1 \otimes V_2$, $\pi\colon G \to \mathrm{GL}(V_1 \otimes V_2) : g \mapsto \pi_1(g) \otimes \pi_2(g)$. This way the representations (up to isomorphism) of a group form a semi-ring (半环, addition has no inverses either). The other operations of vector spaces also generalize to representations, e.g. $\bigwedge^p \pi\colon G \to \mathrm{GL}(\bigwedge^p V) : g \mapsto \bigwedge^p \pi(g)$, and the like.

## Decomposition into irreducible representations I

Given a real (complex) representation $\pi\colon G \to \mathrm{GL}(V)$ with a positive definite bilinear (sesquilinear) form $\langle .,.\rangle\colon V \otimes V \to F$ ($F = \mathbb{R}$ or $\mathbb{C}$), we can average the inner product as

$$\langle\!\langle v, w\rangle\!\rangle := \frac{1}{|G|}\sum_{g\in G}\langle \pi(g)v, \pi(g)w\rangle$$

(note that $\langle\!\langle .,.\rangle\!\rangle$ is still a positive definite inner product) and observe that now $\pi\colon G \to \mathrm{O}(V, \langle\!\langle .,.\rangle\!\rangle)$ (or $\mathrm{U}(V, \langle\!\langle .,.\rangle\!\rangle)$ ), i.e.

$$\begin{aligned}\langle\!\langle \pi(g)u, \pi(g)v\rangle\!\rangle &= \frac{1}{|G|}\sum_{h\in G}\langle \pi(h)\pi(g)u, \pi(h)\pi(g)v\rangle \\ &= \frac{1}{|G|}\sum_{k\in G}\langle \pi(k)u, \pi(k)v\rangle = \langle\!\langle u, v\rangle\!\rangle.\end{aligned}$$

# Decomposition into irreducible representations II

In the case of fields with characteristic nonzero (i.e. $1/|G|$ might not be in $F$) a representation with an invariant inner product is also fully reducible, because of the following fact.

**Lemma**

*Given an invariant subspace $U \subset V$ of a representation $\pi \colon G \to \mathrm{O}(V)$, i.e. $\pi(g)U \subset U$ for all $g \in G$, with an invariant inner product. Then $V$ decomposes into two representations $V = U \oplus U^{\perp}$.*

# Decomposition into irreducible representations III

Proof.

The only thing that needs to be shown is that $U^{\perp}$ is also invariant under $G$. Note therefore the definition of $U^{\perp} = \{v \in V : \langle U, v\rangle = 0\}$ and let $v \in U^{\perp}$ be such a vector and $g \in G$. Then

$$\langle U, \pi(g)v\rangle = \langle \pi(g^{-1})U, v\rangle \subset \langle U, v\rangle = \{0\},$$

therefore also $\pi(g)v \in U^{\perp}$ and thus $\pi(g)U^{\perp} \subset U^{\perp}$. Since $\pi(g)$ is invertible and orthogonal also $\pi(g)|U^{\perp}$ is invertible and orthogonal. This completes the proof. □

Therefore fully reducible representations split uniquely into direct sums of irreducible representations.

# Intertwiners (交结映射) I

An interesting question is how to check whether a representation is irreducible. This can be decided with the following notion.

## Definition

*Given two representations* $\pi_i \colon G \to \mathrm{GL}(V_i)$. *An* equivariant map (等变映射) *is a linear map* $T \colon V_1 \to V_2$ *such that* $T \circ \pi_1(g) = \pi_2(g) \circ T$ *for all* $g \in G$.
*An* intertwiner (交结映射) *is an invertible equivariant map. (In particular the vector spaces must have the same dimension.)*
*A* selfintertwiner (自交结映射) *of a representation is an intertwiner of the representation with itself.*

# Intertwiners (交结映射) II

## Example

0. Given any representation $\pi\colon G \to \mathrm{O}(V)$, then $\lambda \in \mathbb{F}^*$ gives the trivial self-intertwiners $\lambda\mathbb{1}$.
1. Given a depcomposition into invariant subspaces $U_i \subset V$, $V = U_1 \oplus U_2$ of a representation $\pi\colon G \to \mathrm{O}(V)$, then $\lambda_i \in \mathbb{F}^*$ gives the (nontrivial) self-intertwiners $T\colon V \to V : u_1 \oplus u_2 \mapsto \lambda_1 u_1 \oplus \lambda_2 u_2$.

The statement is now named after I. Schur[1] and reads as follows.

## Proposition (Schur's lemma, 舒尔引理)

*Given a representation $\pi\colon G \to \mathrm{O}(V)$, then it is irreducible iff all self-intertwiners are multiples of the identity $\lambda\mathbb{1}$.*

[1] ∗1/1875 in Mogliev †1/1941

## Proof.

In one direction, let $U \subset V$ be an invariant subspace. Therefore $V = U \oplus U^{\perp}$ and thus we can construct the non-trivial intertwiner $T$ as in the previous Example.

Let conversely $T \colon V \to V$ be a non-trivial self-intertwiner. Since the representation is orthogonal, its adjoint $T^{\dagger}$ is also a self-intertwiner. By averaging we can assume that $T$ is self-adjoint. Therefore we can diagonalize it with eigenvalues $\{\lambda_k : k = 1, \ldots, n\}$ where there must be at least one eigenvalue. If there were only one eigenvalue, then $T$ would be a scalar, thus a trivial intertwiner. Therefore there must be at least 2 different eigenvalues. To each eigenvalue there is a $T$-invariant eigenspace $V = U_1 \oplus U_2 \oplus \ldots$ containing at least one nonzero eigenvector each. Since $T$ intertwines with the representation, the representation preserves the eigenspaces (otherwise the eigenvalue of a (generalized) eigenvector could change). Therefore we have an invariant subspace $U_1$ which is nontrivial. Together with the last lemma this gives a decomposition of $V$. This completes the proof. □

### Example

Given the (left)-regular representation $\lambda$ of a finite group $G$, then this has an invariant subspace $V_1 := \langle \sum_{g \in G} e_g \rangle$. This corresponds to the trivial representation $1\colon G \to \mathrm{GL}(F) : g \mapsto \mathbb{1}$.

# Character of a representation

**Definition**

*Given a finite dimensional representation* $\pi: G \to \mathrm{GL}(V)$, *we define its character (特征标)* $\chi: G \to F : g \mapsto \operatorname{tr}\pi(g)$.

**Proposition**

*A character is a class function, i.e.* $\chi(ghg^{-1}) = \chi(h)$ *for all* $g, h \in G$. *And the character of the transpose/ adjoint representation is* $\chi_{\pi^T}(g) = \chi(g^{-1})$. □

# Character of a representation II

## Example

Consider the following representations of $S_3$

| $\pi$ | $\chi$ : id | (12) | (123) | $\lvert\chi\rvert^2$ | comment |
|---|---|---|---|---|---|
| $\lambda$ | 6 | 0 | 0 | 6 | reg. repr. |
| 1 | 1 | 1 | 1 | 1 | $\langle\lambda, 1\rangle = 1$ |
| sgn | 1 | -1 | 1 | 1 | $\langle\lambda, \text{sgn}\rangle = 1$ |
| $\pi_3$ | 3 | 1 | 0 | 2 | fund. repr. |
| $\pi_2 := \pi_3 \ominus 1$ | 2 | 0 | -1 | 1 | $\langle\lambda, \pi_2\rangle = 2$ |

And therefore $\lambda = 1 \oplus \text{sgn} \oplus \pi_2 \oplus \bar{\pi}_2$.

# Character formula

## Proposition

*Given a representation* $\pi\colon G \to \mathrm{GL}(V)$ *on a vector space over an algebraically closed field*[2] *and its character* $\chi$, *then*

1. $\pi$ *is irreducible iff* $|\chi|^2 = 1$,
2. $\chi_{\pi_1 \oplus \pi_2} = \chi_1 + \chi_2$,
3. $\chi_{\pi_1 \otimes \pi_2} = \chi_1 \cdot \chi_2$,
4. $\chi_1$ *the character of an irreducible representation* $\pi_1\colon G \to \mathrm{GL}(V_1)$, *then* $\pi$*s decomposition into irreducible representations contains exactly* $\langle \chi, \chi_1 \rangle$ *copies of* $\pi_1$.

[2] such as $\mathbb{C}$

## Exercise

Given a finite subgroup $G$ of $\mathrm{GL}(V)$ over a real (or complex) vector space, show that

a. every element $g \in G$ has determinant in $\Omega_*$ the roots of unity,
b. give an element $g \in \mathrm{GL}_2(\mathbb{R})$ of finite order that does not have determinant 1;
c. $G$ is isomorphic to a subgroup of $\mathrm{O}(V)$ (or $\mathrm{U}(V)$ respectively).

## Exercise

Decompose the (left)-regular representation of $S_3 \times C_4$ over the complex numbers $\mathbb{C}$ into irreducible representations.