Abstract Algebra – I Groups (群理论)
1.11 The general linear group (一般线性群)

朱翠影 (ZHU Cuiying) et al

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The general linear group (一般线性群) I

Remember the notion of a vector space (向量空间) V over a field (域) F. The linear maps (线性图) from V to itself can be composed and form thus a semigroup with neutral element (the identity $\mathbb{1}_V$). In order for an endomorphism to be invertible its determinant (行列式) may not vanish. The adjoint formula (伴随逆公式, $(\operatorname{ad} A)A = A \operatorname{ad} A = (\det A)\mathbb{1}$) then shows that indeed all such linear maps are invertible.

Definition

Given a vector space V (over a field F), then the general linear group $\mathrm{GL}(V)$ is the set of all linear maps $\phi\colon V \to V$ with nonvanishing determinant $\det \phi \neq 0$.

Note that given a finite dimensional vector space by choosing a base $\{e_1, \ldots, e_n\}$ this is isomorphic to the standard vector space F^n. The linear maps are now encoded by $n \times n$-matrices with entries in F, denoted $\mathrm{Mat}_n(F)$.

The general linear group (一般线性群) II

The set of all invertible matrices is an open subset, because the condition $\det A \neq 0$ is open. It is therefore reasonable to say that $\mathrm{GL}_n(F)$ as well as $\mathrm{GL}(V)$ has dimension n^2.

important subgroups I

1. $\mathrm{SL}_n(F) := \{g \in \mathrm{GL}_n(F) : \det g = 1\}$ the volume preserving linear transformations (特殊线性群);
2. $\mathrm{PSL}_n(F) := \mathrm{SL}_n(F)/\,\mathrm{cent}$ where the center of $\mathrm{SL}_n(F)$ is $\mathrm{cent} = \{\lambda \in F : \lambda^n = 1\}$. For $n \geq 3$ these groups are simple;
3. $\mathrm{O}(n) := \{g \in \mathrm{GL}_n(F) : g^T g = \mathbb{1}\}$ the orthogonal transformations (正交群), this is a maximal compact subgroup if F is real (i.e. $x^2 \geq 0$ for all $x \in F$), $\mathrm{SO}(n) := \mathrm{SL}_n(F) \cap \mathrm{O}(n)$ the orientation preserving (取向保存) orthogonal transformations,

important subgroups II

4. $\mathrm{U}(n) := \{g \in \mathrm{GL}_n(\mathbb{C}) : gg^\dagger = \mathbb{1}\}$ the unitary transformations (酉群), $\mathrm{U}(n) \subset \mathrm{GL}_n(\mathbb{C})$ is correspondingly maximal compact, $\mathrm{SU}(n) := \mathrm{SL}_n(\mathbb{C}) \cap \mathrm{U}(n)$ the volume-preserving unitary transformations. Note that $\mathrm{U}(1) = \mathbb{S}^1$ the circle. Also $\mathrm{SU}(2) = \mathbb{S}^3$, which can be seen if you expand $gg^\dagger = \mathbb{1}$ for $\mathrm{GL}_2(\mathbb{C}) \ni g = \begin{pmatrix} a + b\mathrm{i} & c + c'\mathrm{i} \\ \ldots & \end{pmatrix}$ as $0 = \ldots$ (thus $c' = 0 = d'$) and $1 = a^2 + b^2 + c^2 + d^2$;
5. $\mathrm{Sp}_n(F) := \{g \in \mathrm{GL}_{2n}(F) : g^T J g = J\}$ where $J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$, the symplectic transformations (辛群);
6. Note that $\mathbb{T}^n := \mathrm{U}(1)^n \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ via diagonal elements. This is a maximal abelian subgroup. All maximal abelian subgroups are conjugate;

important subgroups III

7. $B := \left\{ \begin{pmatrix} * & * & \dots & * \\ 0 & * & *\dots & * \\ 0 & \dots 0 & \ddots & \vdots \\ 0 & \dots & 0 & * \end{pmatrix} \right\} \subset \mathrm{GL}_n(F)$ the upper triangular matrices, the Borel group, a maximally solvable subgroup;
8. $\mathrm{ISO}(n) \cong \mathrm{O}(n) \ltimes \mathbb{R}^n$, i.e. the elements are (R, v) where $R \in \mathrm{O}(n)$ is any isometry that fixes the origin and $v \in \mathbb{R}^n$ is a translation vector. The group operation is $(R, v)(R', v') = (RR', v + Rv')$ (with the neutral element $(\mathbb{1}, 0)$ and the inverse elements $(R, v)^{-1} = (R^{-1}, -R^{-1}v)$). This is called semi-direct product (半直积; it is not the direct product, but almost). Note that $\mathrm{ISO}(n) \subset \mathrm{GL}_{n+1}(\mathbb{R})$.

Polar decomposition

Numerically interesting is the following decomposition:

$$\mathrm{GL}_n(\mathbb{C}) = \mathrm{P}_n(\mathbb{C})\mathrm{U}(n), \quad A = PU$$

where $\mathrm{P}_n(\mathbb{C})$ are the self-adjoint positive definite matrices. In the case $n = 1$ this reduces to $z = r\mathrm{e}^{\mathrm{i}\phi}$, i.e. the representation of a (nonzero) complex number in trigonometric form ($r > 0$ and $\mathrm{e}^{\mathrm{i}\phi} \in \mathrm{U}(1)$). The general case is called *polar decomposition (极分解)*. Analytically you can define $P := \sqrt{A^\dagger A}$ (where $A^\dagger$ is the adjoint matrix and thus $A^\dagger A$ self-adjoint and positive (semi)-definite) and $U := P^{-1}A$ (if A and thus P are invertible).