Abstract Algebra – I Groups (群理论)
1.10 Small groups (分类的小群)

曹宁 (CAO Ning) et al

November 12, 2012

Small groups (分类的小群)

Given the structure theory we have been doing so far, you should try to work out a classification of all finite groups of order less than 60.

With Lagrange's theorem we know that every group of prime order is cyclic.

With Sylow's theorems we know that every group G of order pq where $p > q$ and $q \nmid (p-1)$ is abelian and hence cyclic, because $n_p \equiv 1 \pmod{p}$ and $n_p | q$ implies $n_p = 1$ as well as $n_q \equiv 1 \pmod{q}$ and $n_q | p$ implies $n_q = 1$ which are both normal and hence $G \cong \mathbb{Z}/(p) \times \mathbb{Z}/(q) \cong \mathbb{Z}/(pq)$.

Proposition

A group of order $2p$ where p is a prime is either abelian or dihedral.

Proof.

Let G be the group in question. Remember that a group of order less than 6 is abelian. Let thus $p > 2$. We know that $n_p | 2$ and $n_p \equiv 1 \pmod{p}$ hence there is only one p-Sylow subgroup and it is of order p and normal. Conversely there are $n_2 | p$ and $n_2 \equiv 1 \pmod{2}$, 2-Sylow subgroups which are of order 2 each. Since $|G| = 2p = |\mathbb{Z}/(2)| \, |\mathbb{Z}/(p)|$ we know that $G = \langle a, b : a^2 = \mathrm{id} = b^p, \ldots \rangle$.

If $n_2 = 1$, we know that also the only subgroup of order 2 is normal and hence $G \cong \mathbb{Z}/(p) \times \mathbb{Z}/(2) \cong \mathbb{Z}/(2p)$. In general we have $aba = b^k$ for some $k \in \mathbb{N}_+$, because the p-Sylow subgroup is normal. From $a^2 = \mathrm{id}$ it also follows that $b = aaba^{-1}a^{-1} = ab^k a^{-1} = (aba^{-1})^k = b^{k^2}$, hence $p | (k^2 - 1)$. Since p is prime either $p | (k - 1)$ or $p | (k + 1)$.

If $p | (k - 1)$, then $aba = b^k = b$ and thus $ab = ba$, i.e. G is abelian. If on the other hand $p | (k + 1)$, then $aba = b^k = b^{-1}$ and thus $G \cong D_p$.

Some cases of p^k I

A bit more tricky is the situation of groups of order 8. We have seen so far that D_4 and Q (the group of orthogonal unit quaternions) are non-abelian and not isomorphic. We now claim that these are all non-abelian possibilities.

Proposition

Given a non-abelian group of order 8, then it is either isomorphic to D_4 or to Q.

Some cases of p^k II

Proof.

Let G be the group in question. No element of G has order 8, because G is not cyclic. Also not every element can have order 1 or 2, because $g^2 = \mathrm{id}$ for all $g \in G$ would imply $gh = h^{-1}g^{-1} = hg$ for all $g, h \in G$, i.e. that G is abelian. Thus there is an element $a \in G$ of order 4 and $A := \langle a \rangle \triangleleft G$, because it has index 2. Therefore the group G is generated by a and any $b \in G \setminus A$, because $A \subsetneq \langle a, b \rangle \subseteq G$. Moreover $b^2 \in A$, because bA has order 2 in G/A. Also $b^2 \neq a^{\pm 1}$ for otherwise b had order 8. Hence $b^2 = \mathrm{id}$ or $b^2 = a^2$. Moreover $bab^{-1} \in A$ has order 4 like a, but $bab^{-1} \neq a$ (otherwise G were abelian), hence $bab^{-1} = a^3 = a^{-1}$.

If $b^2 = 1$, then the defining relations of D_4 hold in G, i.e. there is a homomorphism of D_4 onto G which is an isomorphism, because both groups have order 8. If on the other hand $b^2 = a^2$, then the defining relations of Q hold .... This completes the proof. □

Some cases of p^k III

Remark

The principle for $n = 2^4$ or $n = 3^3$ is the same, i.e. first find all non-abelian examples (3 in the first case and 2? in the second case), and then try to prove that by exploiting the maximum order of an element in G.

Easy cases of $p^m q^n$

The next step are groups of order 12. Beside the abelian ones, we have already seen D_6 and A_4 (which are not isomorphic). After some trying, we may also come up with $T := \langle a, b : a^6 = \mathrm{id}, b^2 = a^3, bab^{-1} = a^{-1} \rangle$. It seems that we cannot come up with further (non-isomorphic) examples. Indeed that can be shown to be correct also by starting with Sylow's theorem(s) providing a candidate for a normal subgroup.

Proposition

Every non-abelian group of order 12 is isomorphic to either D_6, A_4 or T.

Idea of proof.

Let G be the group in question. Since $12 = 3 \cdot 4$ we know that there is a subgroup $P \subset G$ of order 3. Then G acts by left-multiplication on G/P. Since $(G : P) = 4$ this is an action of G on a 4-element set, i.e. a homomorphism from G into S_4. Its kernel $K \triangleleft G$ is a normal subgroup of G. Moreover $K \subset P$, because $gxP = xP$ for all $x \in G$ implies $g \in P$. Therefore $K = 1$ or $K = P$.

If $K = 1$, then G is isomorphic to a subgroup $H \subset S_4$ of order 12. Let $\sigma \in S_4$ be any 3-cycle. Since $(S_4 : H) = 2$, two of 1, σ, σ^2 must be in the same left-coset of H, i.e. $\sigma \in H$ or $\sigma^2 \in H$, thus $\sigma = \sigma^4 \in H$, i.e. all 3-cycles are contained in H. Therefore $A_4 = H \cong G$.

If $P = K \triangleleft G$, then P is the only 3-Sylow subgroup of G. Thus G has only 2 elements of order 3. Thus every $c \in P$ has at most two conjugates and thus its centralizer (under the conjugation action) has order 6 or 12. Hence by Homework **??** there is an element $d \in \mathrm{cent}_G(c)$ of order 2. For $c \neq \mathrm{id}$, $a := cd$ has order 6. Define $A := \langle a \rangle$ and observe that $A \triangleleft G$, because it has index 2.

As in the proof of the last proposition G is generated by a and any $b \in G \setminus A$. Now $bab^{-1} \in A$ has order 6 like a. Moreover $bab^{-1} \neq a$ otherwise G were abelian, hence $bab^{-1} = a^5 = a^{-1}$. Also $b^2 \in A$, because bA has order 2 in G/A. $b^2 \neq a, a^5$, otherwise G were cyclic. Analogously $b^2 \neq a^2, a^4$, because b commutes with b^2 but $ba^2b^{-1} = a^{-2}$ yields $ba^2 = a^4b$. Hence $b^2 = \mathrm{id}$ which leads to D_6 or $b^2 = a^3$ which leads to T. This completes the proof.

Easy cases of $p^m q^n$ III

Remark

The cases $n = 18$ or $n = 28$ are correspondingly. The case $n = 24 = 2^3 \cdot 3$ is a bit harder (more examples, longer proof).

Easy cases of $p^m q^n$ III

Remark

The cases $n = 18$ or $n = 28$ are correspondingly. The case $n = 24 = 2^3 \cdot 3$ is a bit harder (more examples, longer proof).

Decompositions into 3 or more different prime factors

Remark

The smallest case is $n = 2 \cdot 3 \cdot 5 = 30$ and has only one non-abelian example.

Easy cases of $p^m q^n$ III

Remark

The cases $n = 18$ or $n = 28$ are correspondingly. The case $n = 24 = 2^3 \cdot 3$ is a bit harder (more examples, longer proof).

Decompositions into 3 or more different prime factors

Remark

The smallest case is $n = 2 \cdot 3 \cdot 5 = 30$ and has only one non-abelian example.

Remark

The proofs get essentially more complicated from $n = 60 = 2^2 \cdot 3 \cdot 5$ on, because A_5 also has order 60 and no (non-trivial) normal subgroup, i.e. the Sylow theorem(s) do not provide any normal subgroup.

Classification of groups of order up to 15

In summary we have established the following table.

order	type
1,2,3,5,7,11,13,15	cyclic,
4, 9	$\mathbb{Z}/(p^2)$ or $(\mathbb{Z}/(p))^{\times 2}$,
6, 10, 14	cyclic or dihedral,
8	abelian, D_4 or Q,
12	abelian, D_6, A_4, or $T :=$ $\langle a, b : a^6 = \mathrm{id}, b^2 = a^3, bab^{-1} = a^{-1}\rangle$.

Remark

The next groups are of order 16 and the result is similar to the case 8, but there is one more non-abelian example and the classification proof thus longer.