Abstract algebra: Homework 8

Northwestern Polytechnic University

Due on Monday, Dec. 3rd

2.1 Definition and Examples of Rings and Algebras

Exercise 2.1.1 (1P). Let $(A, +, \cdot)$ be any (unital) non-commutative ring with neutral elements 0 (w.r.t. addition) and 1 (w.r.t. multiplication). Show that $0a = 0 = a0$ for every $a \in A$.

Exercise 2.1.2 (1P). Let $(R, +, \cdot)$ be a set with two monoidal operations $+$ and $\cdot$ neither of which need to be commutative, but both are associative, $\cdot$ is distributive over $+$, i.e.

$$(a + b)c = ac + bc,$$
$$a(b + c) = ab + ac,$$

and $+$ has inverse elements $-a \in R$ for every $a \in R$. Show that $(R, +, \cdot)$ is a (non-necessarily commutative) ring, i.e. $+$ is abelian.
Hint: Consider products $(a + b)(c + d)$.

Exercise 2.1.3 (3P).

a(2P). Given an abelian group $(A, +)$. Show that its group-endomorphisms $\mathrm{End}(A, +)$ form a unital non-commutative associative ring.

b(1P). Given any ring $(R, +, \cdot)$. Show that $(R, +, \cdot)$ embeds canonically into $\mathrm{End}(R, +)$.

Exercise 2.1.5 (5P). Let $\alpha \in \mathbb{C}$ be a zero of a non-trivial polynomial over $\mathbb{Z}$.

a(2P). Show that $\mathbb{Z}[\alpha] = \langle 1, \alpha, \alpha^2, \ldots \rangle_{\mathbb{Z}}$ is a ring together with an embedding $e\colon \mathbb{Z} \to \mathbb{Z}[\alpha] : n \mapsto n \cdot 1$.

b(1P). Show that the Gaussian integers $\mathbb{Z}[i] = \{a + b\mathrm{i} : a, b \in \mathbb{Z}\}$ form a ring. Find its units, i.e. those elements $u \in \mathbb{Z}[i]$ that have a multiplicative inverse $v \in \mathbb{Z}[i]$, i.e. $uv = 1 = vu$. Show that these elements form a group (under multiplication).

c(1P). Show that also $\mathbb{Z}[\sqrt{-2}] \subset \mathbb{C}$ form a subring of the complex numbers. Find its units.

d(1P). What happens when $\alpha \in \mathbb{C}$ is transcendental over $\mathbb{Q}$, i.e. not root of any non-trivial polynomial over $\mathbb{Q}$?

Extra Exercise 2.1.6 (2XP). Let R be a commutative ring (not necessarily with unit). Show that $R^1 := \mathbb{Z} \ltimes R \approx \mathbb{Z} \times R$ with operations $(m, a) + (n, b) := (m+n, a+b)$ and $(m, a)(n, b) := (mn, ab+mb+na)$ for $m, n \in \mathbb{Z}$ and $a, b \in R$ where $ma := a + a + \cdots + a$ (m times) for $m \geq 0$ and $-ma = -(ma)$, correspondingly, is a unital ring. What is its multiplicative identity?

2.2 Homomorphisms, Subrings, and Ideals

Exercise 2.2.1 (1P). Show that the union $\bigcup_{n\geq 0} I_n$ of an ascending chain of ideals $0 \subset I_1 \subset I_2 \subset \ldots$ with $I_n \lhd R$ is an ideal.

Exercise 2.2.3 (3P). Given any ring R. Show that the nil radical $\operatorname{nil\,rad} R := \{z \in R : z^n = 0 \text{ for some } n \in \mathbb{N}\}$ is an ideal. What is the nil radical of $\mathbb{Z}$?

Exercise 2.2.4 (3P). Prove the third isomorphism theorem for rings, i.e. given a subring $S \subset R$ and an ideal $I \lhd R$, then $S + I \subset R$ is a subring, $I \cap S \subset S$ is an ideal and $(S+I)/I \cong S/(S \cap I)$. Draw the mapping behavior of all the maps mentioned and constructed during the proof.
Hint: The mapping scheme is the same as in the corresponding isomorphism theorem for groups.

Exercise 2.2.6 (3P). Let $\phi\colon R \to R'$ be a ring homomorphism and $I \lhd R$ an ideal.

a(1P). Assuming that ϕ is surjective, show that $\phi(I) \lhd R'$ is an ideal.

b(1P). Given a surjective ϕ, show that there is a 1:1 correspondence between ideals $\ker\phi \subset I_1 \subset I_2$ with $I_k \lhd R$ and $I_1' \subset I_2'$ with $I_k' \lhd R'$.

Hint: You also have to show uniqueness of $\ker\phi \subset I \lhd R$ with $\phi(I) = I'$ for any fixed $I' \lhd R'$.

c(1P). Give an example where $\phi(I)$ is not an ideal.