Abstract algebra: Homework 7

Northwestern Polytechnic University

Due on Monday, Nov. 26^{th}

1.13 Normal series

Exercise 1.13.1 (5P).

a. Show that D_4 has a normal series where one of the components is not a normal subgroup of D_4.

b. Given normal series for $N \triangleleft G$ and G/N. Show that these can be pieced together to give a normal series of the group G.

c. Let $\{\text{id}\} \triangleleft G_1 \triangleleft G_2 \triangleleft \ldots \triangleleft G_n = G$ be a normal series. Explain how normal series of each factor G_k/G_{k-1} give a refinement of this normal series. What is needed to obtain a composition series?

d. Given composition series of $N \triangleleft G$ and G/N. How can you obtain a composition series for the group G?

Exercise 1.13.2 (2P). If G has a composition series, then every normal subgroup $N \triangleleft G$ and every quotient G/N (by a normal subgroup) has a composition series.
Hint: Show how N appears in a composition series.

Exercise 1.13.3 (3P). Find all composition series of

a. A_4,

b. D_4,

c. D_5.

1.15 Group extensions

Exercise 1.15.1 (5P). Find all group extensions in the following cases

a. of $Q = C_2$ by $N = C_3$,

b. of $Q = C_3$ by $N = C_2$.

c. Which of the extensions $1 \to C_3 \to G \to C_2 \to 1$ are semi-direct products?

Please turn the page.

1.16 Nilpotent groups

Exercise 1.16.1 (5P). Prove the proposition about the upper central series, i.e. a group is nilpotent iff its upper central series $Z_{n+1} := \{z \in G : \forall g \in G : [g, z] \in Z_n\}$ with $Z_0 := \{\mathrm{id}\}$ ends in G.
Hint: Suppose G is nilpotent of length n (i.e. $G_n = 1$), show that $Z_k \supset G_{n-k}$ and thus $Z_n = G_0 = G$. In the other direction show that $Z_n = G$ implies $G_k \subset Z_{n-k}$.