Abstract algebra: Homework 4

Northwestern Polytechnic University

Due on Monday, Nov. 5th

1.6 Krull–Schmidt theorem

Exercise 1.6.1 (2P). Compute the Krull–Schmidt decomposition of the $D_n = \langle \sigma, \tau : \sigma^2 = \mathrm{id} = \tau^n, \sigma\tau\sigma = \tau^{-1} \rangle$.

Exercise 1.6.2 (8P). Compute the Krull–Schmidt decomposition of $\mathrm{GL}_2(\mathbb{F}_p)$ the automorphism group of the vector space $\mathbb{F}_p^2$ where $\mathbb{F}_p$ is the field with $p \in \mathbb{P}$ elements and p a prime.
Hint: If the general case is too hard, you may start with $p = 3$ (which will give you 3 Points).

1.7 Group actions (群作用)

Exercise 1.7.2 (3P). Let G be a group and for $g \in G$ define the **inner automorphism (内自同构)** $c_g \colon G \to G : h \mapsto ghg^{-1}$.

a. Show that the inner automorphisms c_g form a subgroup $\mathrm{Int}(G) \subset \mathrm{Aut}(G)$ isomorphic to $G/\operatorname{cent}(G)$.

b. Show that $\mathrm{Int}(G) \lhd \mathrm{Aut}(G)$, i.e. a normal subgroup.

Hint: What is $(\phi \circ c_g)(h)$ for $g, h \in G$?

Exercise 1.7.3 (2P). Let $\mu \colon G \times X \to X$ be the action of a group G on a set X.

a. Let $x, y \in X$ be two points on the same orbit. Show that their stabilizers are conjugate, i.e. there is an element $g \in G$ such that $\mathrm{Stab}_G(x) = g\,\mathrm{Stab}_G(y) g^{-1}$.

b. Assume that $\mathrm{Stab}_G(x) \cong C_2$ and $\mathrm{Stab}_G(y) \cong C_3$. Can x and y be on the same orbit? (Justify your answer.)

Exercise 1.7.4 (2P). Show that in a finite group G of order n, an element of order k has at most n/k conjugates.

Exercise 1.7.5 (3P). Determine the class equation of each $D_n := \langle \sigma, \tau : \sigma^2 = \mathrm{id} = \tau^n, \sigma\tau\sigma^{-1} = \tau^{-1} \rangle$, $n = 1, 2, \ldots$.