Abstract algebra: Homework 3

Northwestern Polytechnic University

Due on Monday, Oct. 29th

1.5 Direct products

Exercise 1.5.1 (Product and coproduct, 直积与对偶直积; 5P). Given two groups G and H.

a. Show that their direct product $G \times H$ together with the canonical projections $\pi_G \colon G \times H \to G$ and $\pi_H \colon G \times H \to H$ is a product (in the sense of category theory), i.e. for every pair of homomorphisms $\rho_G \colon K \to G$ and $\rho_H \colon K \to H$ there is a unique morphism $\tilde{\rho} \colon K \to G \times H$ such that $\pi_G \circ \tilde{\rho} = \rho_G$ and $\pi_H \circ \tilde{\rho} = \rho_H$.

b. Show that $G \oplus H = G \times H$ together with the canonical embeddings $e_G \colon G \to G \oplus H$ and $e_H \colon H \to G \oplus H$ is a coproduct, i.e. for every pair of homomorphisms $\rho_G \colon G \to K$ and $\rho_H \colon H \to K$ with $\rho_G(g)\rho_H(h) = \rho_H(h)\rho_G(g)$ for all $g \in G$ and $h \in H$, there is a unique morphism $\tilde{\rho} \colon G \oplus H \to K$ such that $\rho_G = \tilde{\rho} \circ e_G$ and $\rho_H = \tilde{\rho} \circ e_H$.

c (1P)**.** Draw the mapping behavior of the morphisms in Parts a and b. You may use solid arrows for given morphisms and dashed arrows for morphisms implied during your proof.

Exercise 1.5.2 (2P). Find all abelian groups (up to isomorphism) of order

a. 35,

c. 360.

Exercise 1.5.4 (3P). A group G is called **indecomposable** iff for every direct sum $G \cong A \oplus B$ either $A = 1$ or $B = 1$.

a. Prove that D_5 is indecomposable;

b. prove that D_4 is indecomposable;

Please turn the page.

c. prove that C_{p^k} is indecomposable when p is a prime and $k \in \mathbb{N}$.

1.6 Krull–Schmidt theorem

Exercise 1.6.1 (2P). Compute the Krull–Schmidt decomposition of the $D_n = \langle \sigma, \tau : \sigma^2 = \mathrm{id} = \tau^n, \sigma\tau\sigma = \tau^{-1} \rangle$.

Exercise 1.6.2 (8P). Compute the Krull–Schmidt decomposition of $\mathrm{GL}_2(\mathbb{F}_p)$ the automorphism group of the vector space $\mathbb{F}_p^2$ where $\mathbb{F}_p$ is the field with $p \in \mathbb{P}$ elements and p a prime.
Hint: If the general case is too hard, you may start with $p = 3$ (which will give you 3 Points).