Abstract algebra: Homework 2

Northwestern Polytechnic University

Due on Monday, Oct. 22nd

2.3 Isomorphism theorems

Exercise 2.3.1 (2P). Let $\phi\colon A \to B$ and $\psi\colon A \to C$ be group homomorphisms. Prove the following: If ψ is surjective, then ϕ factors through ψ if and only if $\ker\psi \subset \ker\phi$. In this case ϕ factors uniquely through ψ.

Exercise 2.3.2 (1P). Show that the identity homomorphism $\mathrm{Id}\colon 2\mathbb{Z} \xrightarrow{\sim} 2\mathbb{Z}$ does not factor through the inclusion homomorphism $\imath\colon 2\mathbb{Z} \hookrightarrow \mathbb{Z}$ (there is no $\phi\colon \mathbb{Z} \to 2\mathbb{Z}$ such that $\mathrm{Id} = \phi \circ \imath$) even though $\ker\imath \subset \ker\mathrm{Id}$.

Hint: Opposite to the situation in Exercise 2.3.1, $\imath$ is not surjective.

Exercise 2.3.5 (2P).

a. Show that the additive group $\mathbb{R}/\mathbb{Z}$ is isomorphic to the multiplicative group of all complex numbers $\mathbb{C}$ of modulus 1.

b. Show that the additive group $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the group of all complex roots of unity (i.e. all complex numbers $z \neq 0$ such that $\langle z\rangle$ is finite in $\mathbb{C}^*$).

c. Show that the complex n-th roots of unity $\Omega_n := \{z \in \mathbb{C} : z^n = 1\}$ form a cyclic group (w.r.t. multiplication).

Exercise 2.3.6 (2P). Consider the group $D_4 := \langle \sigma, \tau : \sigma^2 = \mathrm{id} = \tau^4, \sigma\tau\sigma = \tau^{-1}\rangle$

a. Find the order of every element in D_4,

b. Show that for every $d | (D_4 : 1)$ there is a subgroup $S \subset D_4$ of order d.

Exercise 2.3.7 (3P).

a. Let G be a finite group and $S, T \subset G$ any subgroups. Show that $|ST| = |S|\,|T|/|S \cap T|$.

b. Find a group G together with subgroups $S, T \subset G$ such that $ST \subset G$ is not a group.

Please turn the page.

2.4 Free groups, free products, and presentations

Exercise 2.4.1 (4P). Given a group G, the conjugates of an element $x \in G$ are $C_x := \{gxg^{-1} : g \in G\}$. Given a subset $S \subset G$, there exists a smallest normal subgroup $N \lhd G$ that contains $S \subset N$. Show that N consists of all products of elements in $C_{S \cup S^{-1}}$.

Exercise 2.4.2 (3P).

a. List (compactly) all elements of the group $\langle a, b : a^2 = \mathrm{id} = b^2 \rangle$. Give a compact multiplication table of the group.

b. List all elements of the group $\langle a, b : a^2 = \mathrm{id} = b^2 = (ab)^3 \rangle$ and give their multiplication table. Which known group is it isomorphic to?

Exercise 2.4.3 (3P). The multiplication of the unit quaternions $i^2 = -1 = j^2 = k^2$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$ together with $\mathbb{R}$-linearity implies for $a, b, c, d, a', b', c', d' \in \mathbb{R}$,

$$\begin{aligned}(a + bi + cj + dk)&(a' + b'i + c'j + d'k) = \\ &= (aa' - bb' - cc' - dd') + (ab' + ba' + cd' - dc')i \\ &\quad + (ac' + ca' + db' - bd')j + (ad' + da' + bc' - cb')k.\end{aligned} \tag{1}$$

a. Show that the multiplication is associative.

b. Let $\overline{a + bi + cj + dk} := a - bi - cj - dk$ and $|z|^2 := z\bar{z}$ for every quaternion $z \in \mathbb{H}$. Show that $|z_1 z_2| = |z_1||z_2|$ for every pair of quaternions $z_{1/2} \in \mathbb{H}$.

c. Conclude that $\mathbb{H}\backslash\{0\}$ is a group under multiplication. (What is the inverse? Therefore $\mathbb{H}$ is called a division algebra.)