Abstract algebra: Homework 12

Northwestern Polytechnic University

Due on Wednesday, Jan. 2nd

3.4 Resultants & Discriminants

Exercise 3.4.1 (2P). When do $x^2 + ax + b$ and $x^2 + px + q \in F[x]$ have common roots?

Exercise 3.4.2 (3P). Verify the formula for the discriminant of $x^4 + rx^2 + sx + t$.

Exercise 3.4.3 (2P). Write the symmetric function $p_d(x_1, \ldots, x_n) := x_1^d + \cdots + x_n^d \in F[x_1, \ldots, x_n]$ as a polynomial $\bar{f} \in F[s_1, \ldots, s_n]$ in the elementary symmetric functions

b. for $d = 3$,

c. for arbitrary $d \in \mathbb{N}$.

Exercise 3.4.4 (1P). We know that for quadratic polynomials $x^2+px+q \in \mathbb{R}[x]$ the polynomial factors over $\mathbb{R}$ iff the discriminant $D := p^2 - 4q$ fulfills $D \geq 0$. What is the corresponding condition for quadratic polynomials over arbitrary fields F not of characteristic 2?

3.5 Splitting fields and Normal extensions (正规扩张)

Exercise 3.5.1 (2P). Find counter examples for the Remark 3.5.5, i.e.

0. a normal field extension $F \subset E$ together with an intermediate field $F \subset K \subset E$ such that K/F is not normal;

a. two normal field extensions $F \subset K$ and $K \subset E$ such that E/F is not normal.

Exercise 3.5.2 (Structure of finite fields, 2P). Show that $\mathbb{F}_{p^m}$ and $\mathbb{F}_{p^n}$ are embedded in $\mathbb{F}_{p^l}$ with $l = \operatorname{lcm}(m, n)$ and their intersection (in the embedding) is $\mathbb{F}_{p^d}$ with $d = \gcd(m, n)$.

Remark. This implies that $\bar{\mathbb{F}}_p$ is the inductive limit $\bar{\mathbb{F}}_p = \varinjlim_n \mathbb{F}_{p^n}$. Note however that the embeddings $\mathbb{F}_{p^m} \hookrightarrow \mathbb{F}_{p^n}$ exist only for certain m and n, i.e. this is still a directed system (with a partial order), but not a linear chain.

Please turn the page.

Exercise 3.5.3 (2P). Consider a field $F \subset \bar{F}$ together with a family of intermediate fields $F \subset E_\alpha \subset \bar{F}$ and prove the following:

a. if all E_α are normal over F, then so is their intersection;

b. the normal closure of an algebraic extension $F \subset E \subset \bar{F}$ is the composite of all conjugates of E, i.e. the images of E under all F-automorphisms of $\bar{F}$.

3.6 Galois extensions (伽罗瓦扩张) and the correspondence principle (对应原理)

Exercise 3.6.1 (3P). Let $F \subset E$ be a finite Galois extension and consider intermediate fields $F \subset K_i \subset E$ as well as corresponding subgroups $H_i \subset \mathrm{Gal}(E : F)$. Prove the following:

a. $K_1 \subset K_2$ iff $H_1 \supset H_2$;

b. $K_1 = K_2K_3$ iff $H_1 = H_2 \cap H_3$;

c. $K_1 = K_2 \cap K_3$ iff $H_1 = \langle H_2, H_3 \rangle_{\mathrm{Gal}}$.

Extra Exercise 3.6.2 (Galois connection, 2XP). Given two partially ordered sets $(X, \leq)$ and $(Y, \leq)$ together with order reversing maps $F\colon X \to Y$ and $G\colon Y \to X$, i.e. for all $x_i \in X$ with $x_1 \leq x_2$ there is $F(x_2) \leq F(x_1)$ and for all $y_i \in Y$ with $y_1 \leq y_2$ we have $G(y_2) \leq G(y_1)$. Show that F and G induce mutually inverse order-reversing bijections between $X^{GF} := \{x \in X : (GF)(x) = x\}$ and $Y^{FG} := \{y \in Y : (FG)(y) = y\}$.

Remark. The pair (F, G) is called a Galois connection between X and Y.

Exercise 3.6.3 (3P). Let $p = x^3 + x - 1 \in \mathbb{Q}[x]$ and compute its Galois group together with all intermediate fields between $\mathbb{Q}$ and the splitting field E of p. Which are Galois extensions over $\mathbb{Q}$?