Abstract algebra: Homework 11

Northwestern Polytechnic University

Due on Monday, Dec. 24^{th}

3.1 Algebraic and Transcendental Extensions

Exercise 3.1.1 (2P).

a. Give a short proof to show that there is no field of order 6.

c. Given the example $\mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}(x)/(x^2-2)$ of degree 2 over $\mathbb{Q}$, what would you need to construct a field with 4, 8, 9, 16 elements?

Exercise 3.1.4 (4P). Assuming Proposition 3.1.10, show Corollary 3.1.11, i.e. Given a field extension $F \subset E$, a subset $S \subset E$, and some elements $r, \alpha_1, \ldots, \alpha_n \in E$, then

a. $r \in F[\alpha_1, \ldots, \alpha_n]$ iff there is some polynomial in n indeterminates $p \in F[x_1, \ldots, x_n]$ such that $r = p(\alpha_1, \ldots, \alpha_n)$;

b. $r \in F(\alpha_1, \ldots, \alpha_n)$ iff there is some rational function in n indeterminates $f \in F(x_1, \ldots, x_n)$ such that $r = f(\alpha_1, \ldots, \alpha_n)$;

c. $r \in F[S]$ iff there are some $\alpha_1, \ldots, \alpha_n \in S$ such that $r \in F[\alpha_1, \ldots, \alpha_n]$;

d. $r \in F(S)$ iff there are some $\alpha_1, \ldots, \alpha_n \in S$ such that $r \in F(\alpha_1, \ldots, \alpha_n)$.

3.2 Algebraic closure

Exercise 3.2.1 (2P). Suppose that a, b are algebraic over the field F (with minimal polynomials) of degree m and n, respectively. What can you say about the degree (of the minimal polynomials) of $a \pm b$, ab, a/b (for $b \neq 0$)?

Exercise 3.2.2 (3P). Show that every algebraically closed field is infinite. *Hint:* You can assume that over every $\mathbb{F}_p$ ($p \in \mathbb{P}$ a prime) and every positive integer $n \in \mathbb{N}_+$ there is (at least one) an irreducible polynomial of degree n.

Please turn the page.

3.3 Separable extensions

Exercise 3.3.2 (3P). Let $F \subset E$ be any algebraic extension. Show that the set $\mathrm{Sep}(E/F) := \{\alpha \in E : \alpha \text{ separable over } F\}$ is a subfield of E.

Exercise 3.3.4 (2P). Prove the following tower property of purely inseparable field extensions: Given a tower of algebraic field extensions $F \subset K \subset E$, then E/F is purely inseparable iff E/K and K/F are purely inseparable.

Exercise 3.3.5 (4P). Show the following properties of the **inseparability degree (纯不可分度数)** of an algebraic extension $F \subset E$ of finite degree $[E : F]_i := [E : F]/[E : F]_s$:

a. $[E : F]_i \in \mathbb{N}_+$,

b. if $[E : F]_i \geq 2$, then there is at least one purely inseparable element $\alpha \in E$;

c. if $F \subset K \subset E$ are finite algebraic extensions, then $[E : F]_i = [E : K]_i[K : F]_i$.