Abstract algebra: Homework 10

Northwestern Polytechnic University

Due on Monday, Dec. 17th

2.5 Unique factorization domains

Exercise 2.5.1 (3P). Show that every family of elements $a_i \in R$ of a UFD has a

a. greatest common divisor;

b. least common multiple if the family is finite.

c. Show that there is a UFD with an infinite family of elements that does not have a finite non-zero least common multiple.

Hint: You cannot assume that a UFD is a PID, but nevertheless the gcd is determined uniquely by the common irreducible divisors modulo equivalence, while the lcm is determined by the union of all irreducible divisors modulo equivalence.

Exercise 2.5.2 (2P). Find a UFD R together with two elements $a, b \in R$ such that their greatest common divisor cannot be written as a linear combination $\gcd(a, b) = fa + gb$ for any $f, g \in R$.

Exercise 2.5.3 (2P). Prove the following: Assume R is a UFD, $I \lhd R$ any ideal and $\pi \colon R \to R/I$ the quotient map. If $f \in R[x]$ monic and $f_\pi \in (R/I)[x]$ irreducible, then f is irreducible over R.

Exercise 2.5.4 (3P). Apply the previous result to show that the following polynomials are irreducible in $\mathbb{Q}[x]$:

a. $x^3 - 10$,

b. $x^3 + 3x^2 - 6x + 3$,

c. $x^3 + 3x^2 - 6x + 9$,

Please turn the page.

2.8 Localizations

Exercise 2.8.1 (5P). Show that the polynomial $Q = x^4 + 4x^3 + 5x^2 + 1 \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Q}[x]$. You may proceed as follows:

0. Note that you cannot apply Eistenstein's criterion directly;

a. localize the polynomial w.r.t. $p = 2$ and note that it factors. This says, that one localization is not sufficient;

b. localize the polynomial w.r.t. $p = 3$ and note that it also factors. Thus this localization is not sufficient either;

c. assume that Q factors in $\mathbb{Q}[x]$ as $Q = fg$ and compare their localizations for $p = 2, 3$ with the factors you obtained earlier. Conclude that either f or g must have degree 4 and thus Q is irreducible.

Exercise 2.8.2 (2P). Given the ring of integers $R = \mathbb{Z}$ and the multiplicative set $S = R \setminus (2)$. Determine the localization $S^{-1}R$ and show that it is embedded into $K[R] = \mathbb{Q}$. What is its image?