Abstract algebra: Homework 1

Northwestern Polytechnic University
Course home page: melchiorG.freehosting.com/algebra

Due on Monday, Oct. 15th

Exercise 1.1.1 (3P). Given a semi-group $(X, \cdot)$ with a left-neutral element (i.e. $e \in X$ such that for all $a \in X$: $ea = a$) and left-inverses (for all $a \in X$ there is an $a_L \in X$ such that $a_L a = e$), show that $(X, \cdot)$ is a group.

What happens when we require a right-neutral and right-inverse elements?

Extra Exercise 1.1.3 (3XP). * Let $(X, \cdot)$ be a finite semi-group. Assume that for every $a \in X$ the cancellation law holds, i.e. $ab = ac$ implies $b = c$ and $ba = ca$ implies $b = c$. Show that $(X, \cdot)$ is a group.

Given a example of an infinite semigroup where the cancellation law holds, but that is not a group.

Exercise 1.1.4 (2P). Describe the group of symmetries of the sine curve ($y = \sin x$ over the real numbers,), i.e. list all its elements and write a multiplication table (compactly).

Exercise 1.1.5 (2P). Given a group $(G, \cdot)$. Show that $a^m a^n = a^{m+n}$ for all $a \in G$ and $m, n \in \mathbb{Z}$ where a^m has the usual meaning, i.e. $a^m = \underbrace{aa \cdots a}_{m \text{ times}}$ for $m > 0$, $a^0 = \text{id}$ and $a^{-m} = (a^m)^{-1}$. Show moreover $(a^m)^n = a^{mn}$ for the same elements.

Exercise 1.1.6 (3P). Prove that a finite group with an even number of elements contains an even number of elements x such that $x^{-1} = x$.

State and prove a similar statement for finite groups with an odd number of elements.

Please turn the page.

*These are more difficult, you may try them and if you solve them (partially), you get (partial) extra credit. If you don't solve them, you don't lose points. They will not be required for the final exam.

1.2 Group homomorphisms, subgroups, and ideals

Exercise 1.2.1 (4P). Given a group G and a family of subgroups $\{S_\alpha \subset G : \alpha \in A\}$. Show that

a. The intersection $\bigcap_{\alpha \in A} S_\alpha$ is a subgroup;

b. if all $S_\alpha \lhd G$ are normal, then the intersection is also normal.

Exercise 1.2.4 (1P). Show that every group of prime order is simple (i.e. has only trivial subgroups) and cyclic.

Exercise 1.2.5 (5P).

a. Let G be a group generated by $X \subset G$. Prove that for two homomorphisms $\phi, \psi \colon G \to H$ into any group H, $\phi(x) = \psi(x)$ for all $x \in X$ is equivalent to $\phi = \psi$.

b. Find all endomorphisms of $V_4 := \langle (12)(34), (13)(24), (14)(23) \rangle \subset S_4$ (Klein's four group).

c. Find all automorphisms of V_4.

d. Find all endomorphisms and automorphisms for D_3.